ADVANCES AND PERSPECTIVES IN THE TEACHING OF MATHEMATICAL MODELLING AND APPLICATIONS

SLOYER, BLUM AND HUNTLEY

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Advances and Perspectives in the Teaching of Mathematical Modelling and Applications

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Preface

The teaching of mathematical modelling and applications is here. For more then 20 years, groups involved in mathematics education all over the world have initiated, supported, and directed the inclusion of modelling and applications into the curriculum at all levels. Writing groups, on an international basis, have prepared materials which use actual applications of mathematics as a catalyst for educational reform. The power of modelling has enabled educators to focus on real–world problems and the development of responsible citizenship. References to support these statements abound in this book.

Now that a vanguard group has prepared the way, what advances have been made and where are we heading? These questions form a focus for this book. The introduction of mathematical modelling and applications has introduced a need for new teaching methods and classroom organization, but there is still a great deal to be done as new issues are raised. For example, the role of computers in mathematics education and the broad spectrum of mathematical topics used in modelling present problems educators have not yet solved completely. Perspectives on such problems are treated in this book.

This volume is divided into six sections. Section A consists of three chapters, each giving a **survey** of advances and perspectives in mathematical modelling in an educational context. Section B deals with **empirical investigations** involving a modelling process. Section C looks at the important issue of **assessment**. Section D treats the **industrial collaboration** so necessary for the procurement of realistic problems. Section E deals with **primary and secondary examples** and, finally, section F with **tertiary examples**.

The chapters in this book represent the proceedings of the 6th International Conference on the Teaching of Mathematical Modelling and Applications, ICTMA-6, held at the University of Delaware, USA, in August 1993. This was one in a series of biennial international conferences concerned with the teaching of modelling and applications. The first two conferences were held in Exeter, UK (in 1983 and 1985), the third in Kassel, Germany (1987), the fourth in Roskilde, Denmark (1989), and the fifth in Utrecht, The Netherlands (1991). The next conference, ICTMA-7, will take place in July 1995 in Jordantown, Northern Ireland (UK).

The editors wish to thank the authors for their excellent contributions to the present book. They would also like to extend their gratitude to Diane Klonowski for her help and patience in the preparation of the manuscript.

Cliff Sloyer, Werner Blum, Ian Huntley

I would like to thank Werner Blum and Ian Huntley for their aid in editing the manuscript. Any errors or inconsistencies which remain are entirely my responsibility.

Cliff Sloyer

Section A
Surveys

1

Applications and Modelling in Mathematics Teaching and Mathematics Education – Some Important Aspects of Practice and of Research

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SUMMARY

This paper will address some essential questions concerning the theory and practice of mathematical modelling and applications in learning and teaching mathematics at the secondary and tertiary levels. It will consist of six parts. In part 1, basic notions will be clarified by means of an example. Part 2 will review briefly essential arguments for the inclusion of applications and modelling in mathematics teaching at schools and universities. Part 3 will describe the role of applications and modelling in present mathematics curricula and in everyday teaching practice, including some difficulties and obstacles. In part 4, selected recent resources and materials for teaching mathematical modelling and applications will be referenced. Part 5 will analyse the opportunities and risks of using computers in this context. In part 6, general measures for overcoming the aforementioned difficulties and obstacles will be suggested, as well as a research programme for applications and modelling in learning and teaching mathematics.

1. WHAT DOES APPLICATIONS AND MODELLING MEAN? EXAMPLES AND NOTIONS

In the literature, there are a lot of different definitions of terms such as 'application' or 'modelling'. Therefore it is reasonable to start with defining briefly and pragmatically my terminology. I will do this with the aid of an example (in two parts) which I shall refer to several times later on. This example is not new though some features of the following presentation may be so; compare with Winter (1975) and Burghes, Huntley and McDonald (1982). It is not meant to be 'representative' in any sense; it is simply an appropriate case for my didactical intentions. Of course, I can only outline the essential steps.

EXAMPLE: TRAFFIC FLOW

Part 1: Traffic lights

Situation: A road junction with traffic lights.
Problem: How to install and control traffic lights so that the traffic streams as fluidly and as safely as possible?
Structurisation: We decide on 9 lights according to the 9 streams of traffic.

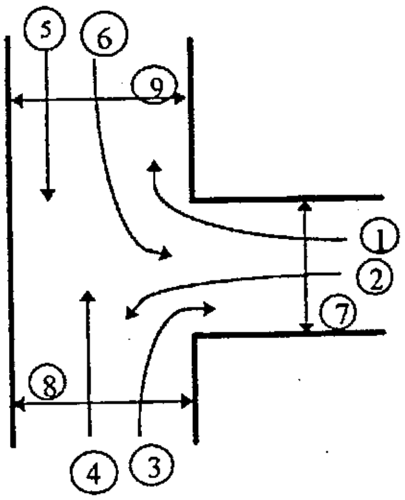

Fig. 1

Question: Which streams are compatible with one another?
Mathematisation: Compatibility graph as a mathematical model.

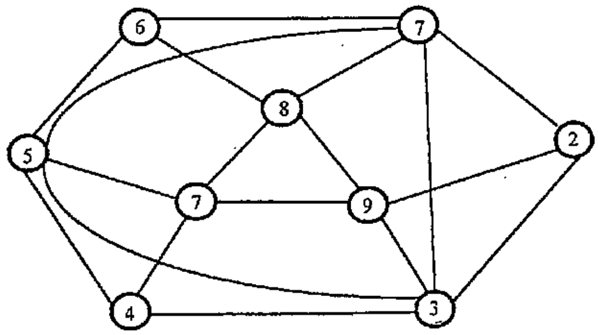

Fig. 2

Mathematical problem: Find sets of maximally complete subgraphs with each vertex occurring at least once.
Solution: For instance

$$\{\langle 7,8,9\rangle, \langle 1,6,8\rangle, \langle 1,2,3\rangle, \langle 3,4,5\rangle\}.$$

Interpretation: Control the lights according to this set, in the given order of phases.
Is this solution to the original problem "optimal" with respect to our intentions? So we had better find more or even all solutions and compare these.
Refinement: We assign times to the individual light phases, dependent on the amount of traffic and on political intentions.

Part 2: Traffic flow rate

Situation: A single-lane road with dense motor traffic.
Problem: At what speed should cars go in order to maximise flow rate?

Simplification:
- All cars at same constant speed (v)
- All cars of same length (l)
- Same distance (d) between cars

We imagine a fixed registering point and define *flow rate* (F) as the number of cars per time at this point.

Mathematisation: Formula $F = \frac{v}{l+d}$ as a **mathematical model** .
d depends on $v : d = g(v)$. Let $v =< v >$, where $< v >$ indicates the speed reading in km/hr.
Model 1: Half-speed rule: $d = \frac{<v>}{2} m = g_1(v)$
Model 2: 1.5-second rule: $d = v \cdot 1.5$ sec. $= g_2(v)$

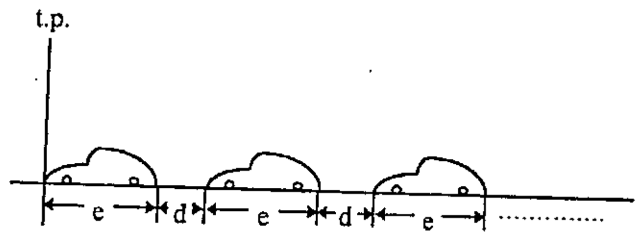

Fig. 3

Model 3: Driving-school rule: $d = \left(3 \cdot \frac{<v>}{10} + \left(\frac{<v>}{10}\right)^2\right) m = g_3(v)$
Model 4: Stopping-distance rule: $d = v \cdot t_R + \frac{1}{2a}v^2 = g_4(v)$
Model 5: Front-man rule: $d = v \cdot t_R + \frac{1}{2}\left(\frac{1}{a_0} - \frac{1}{a_1}\right) v^2 = g_5(v)$

It would be interesting in itself to analyse and compare these rules since most people have fatal misconceptions about stopping distances. I will merely state that models 1 and 2 are (unrealistic) cases of 5 and model 3 is a special case of 4 and of 5.

Flow rate models: $f_i(v) = \frac{v}{l+g_i(v)}$ $(v \geq 0)$
Mathematical problem: Maximise f_i

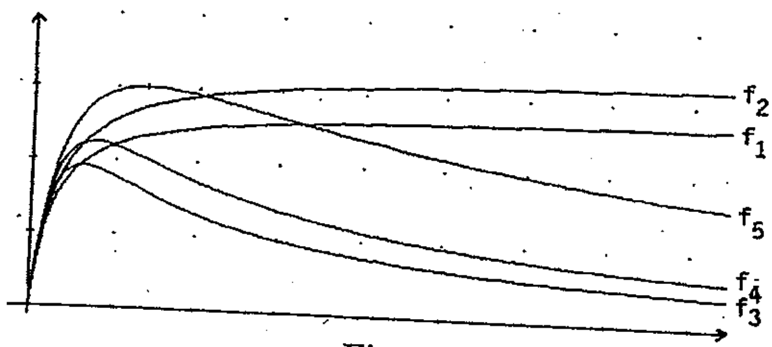

Fig. 4

In the case of f_4, for example, a rough inspection is sufficient for our purposes since the maximum is rather flat. For reasonable values of the parameters we get

$$v \approx 30 \text{ km/hr}.$$

Interpretation: Maximal flow rate according to model 4 if $v \approx 30$km/hr.

That's remarkably low, so passionate motorists won't like the result. If we compare with f_5 for reasonable values we get

$$v \approx 50\text{km/hr}.$$

Again that's quite low. However, because of the very flat maximum it doesn't really matter if cars drive a bit faster, and that might even be better ecologically (provided it's a motor-only road, such as a motorway).

A further mathematical analysis will use differential calculus to determine maxima and yield for model 3, 4, 5 $v_{\text{opt}} = \sqrt{2al}$, that means braking distance equals car length if $v = v_{\text{opt}}$.

Refinement: We consider varying car lengths or reaction times.

Apparently, both parts of the example are structured according to the well-known simple model of *applied mathematical problem-solving*: From a real situation to a real model, then – if possible – to a mathematical model, to mathematical results and then back to the situation. At this point the model is also validated. If discrepancies occur then the whole cycle may start again, or the problem solver may leave it at that, in accordance with JM Keynes' well-known aphorism "It is better to be roughly right than precisely wrong."

Actually, these features fit in only with 'really real' situations. Sometimes – especially in school mathematics – the given situation is just a dressing up of some purely mathematical problem. Then model-building means merely undressing, and the process stops after one cycle only. Nevertheless, such *word problems* – and all kinds of problems in between totally authentic and totally artificial – may quite well be of didactical value and are also included in what follows.

Now to my terminology. A real-world situation can be called an *application*, and any connection between mathematics and reality can be denoted an *application* of mathematics. The term (mathematical) *modelling* may mean the process of model building, leading from a real situation to a mathematical model, or the whole applied problem-solving process, or sometimes any manner of connecting the real world with mathematics. In this paper, modelling means applied problem-solving. As an all-comprising expression for these various meanings, concerning both objects and processes, I will use the composite term *applications and modelling* (abbreviated to A&M; compare also Blum and Niss, 1991).

Much more about the status and the nature of interrelations between mathematics and the real world can be found in the vast literature on that topic, for example in recent books such as Huntley and James (1990), Murthy, Page and Rodin (1990) or Giordano and Weir (1993), and particularly in the proceedings of the ICTMA conferences: Berry *et al.* (1984), Berry *et al.* (1986, 87), Blum *et al.* (1989a), Niss *et al.* (1991), deLange *et al.* (1993), and of the applications and modelling parts of the ICME conferences: Blum *et al.* (1989b), Breiteig *et al.* (1993).

2. WHAT IS THE USE OF APPLICATIONS AND MODELLING IN MATHEMATICS TEACHING? ARGUMENTS AND AIMS

In the last 15 years, there has been a world-wide *trend* towards A&M in mathematics education (see, for example, Blum and Niss, 1991). Exemplary documents are the two books by NCTM (1989, 1991) which contain real-world problem-solving and extra-mathematical connections as essential components in all parts.

This is an encouraging development, so there is no need to advertise for A&M. However, in the educational debate, occasionally only single arguments are put forward. That is why it is even possible to identify certain schools of thought within the maths education community according to the aims and arguments placed in the foreground (*cf.* Kaiser-Messmer, 1991). Yet, the intended aims are closely linked with didactical principles and thus have implications for method and organisation of teaching as well. Therefore it does make sense to summarise, once again, the essential reasons for favouring A&M and to hint at instructional implications. In an international perspective I see four arguments, mainly based on general aims for mathematics education (see Blum, 1991).

Pragmatic arguments Maths instruction is intended to help students to understand and to cope with real-world situations and problems and to prepare them for their future lives as responsible citizens or competent workers in a democratic society of the next century. To that end, suitable applicational examples are indispensable.

Formative arguments By being concerned with mathematics, students should – we hope – acquire general qualifications such as the ability to communicate or cooperate with other people, and general attitudes such as willingness to enter into new situations. Involving students actively in A&M problems is one possible way to develop these.

Cultural arguments Students should be taught mathematical topics as a source for reflection and in order to generate as comprehensive and balanced a picture of mathematics as possible, as a science and as a part of human culture. Linking mathematics to reality, using (or misusing) mathematics, has always been crucial for the history, philosophy and social practice of mathematics. Thus, dealing with applications in the classroom, especially with genuine modelling examples of a more global kind, can contribute towards those aims. This should also comprise critically judging and discussing models, as well as letting students gain insight into the peculiar phenomenon that our society is increasingly mathematised but mathematics becomes more and more hidden, especially by new technologies (see Keitel, 1993). This will help students to develop higher-order knowledge, 'reflective knowledge' in the sense of Skovsmose (1989). The traffic problems, for example, can show how mathematical models are implicitly inherent in daily life through traffic control regulations, or can lead to discussions about the validity and limitations of models.

Psychological arguments Among many other things, applications may contribute toward longer retention of mathematical topics, or change students' attitudes toward maths. In particular, real-world interpretations can support the understanding of mathematical concepts and the formation of basic ideas (such as derivative as rate of change), as well as supply suitable contexts for reasoning mathematically on a non-formal but absolutely rigorous level (for instance, determining monotonicity and maxima in the traffic flow rate problem without using derivatives).

A more general argument is the following. Students often experience mathematics as a mechanical manipulating of meaningless symbols. At best they acquire – philosophically speaking – some dispositional knowledge. A&M is one way to make the learning and teaching of mathematics more meaningful and to supply students with orientational knowledge as well.

All arguments are relevant for all kinds of maths teaching at all levels (secondary and tertiary, general and vocational), though with different emphases for different educational histories (see Blum, 1991). Of course, A&M is only one component in the complex field of learning and teaching mathematics. Stressing this component too much also leads to a reductionistic picture of maths. I think that after the eighties, when A&M was emphasized quite a lot in the didactics of mathematics all over the world (some people even spoke of a fashion wave), we are now in a phase where we are becoming better aware of the place of A&M in that field, where hardly any new specific perspectives are coming into play, where the debate is tending toward a consensus, and issues other than A&M are coming to the fore (for instance, the social dimension of mathematics or the role of computers). This is absolutely right. Nevertheless, there is a certain danger if A&M is no longer a central topic in the educational debate. For, it is then possible that the efforts to improve the actual situation with respect to A&M might also decrease. Yet such efforts are still necessary, there is still a lot to do, especially in teaching practice.

3. WHAT IS THE ROLE OF APPLICATIONS AND MODELLING IN MATHEMATICS CURRICULA AND EVERYDAY TEACHING PRACTICE? TENDENCIES AND DIFFICULTIES

Globally speaking, there has been a world-wide trend in the last decade towards including more A&M components into mathematics syllabi and textbooks. Some present-day secondary school curricula, especially from the Netherlands and from Australia, have included compulsory A&M components throughout.

The actual implementation, however, has not been uniform. There are considerable variations, with respect to

- the aims aspired to,
- the mathematical topics with A&M content,
- the extra-mathematical areas that examples are taken from,
- the proportion of intra- and extra-mathematical contents,
- the conception for the combination of mathematical and applicational components,
- the scope and kind of examples,
- the expected activities of students.

In recent years, the curricular developments in various countries seem to be heading in the right direction; there are trends towards

- broadening the spectrum of aims,
- broadening the range of applied mathematical topics (for example, including more probability and statistics, more discrete mathematics, and new topics such as chaos or fractals),
- broadening the range of applications (examples from economics, ecology, sports or arts in addition to classical physics or everyday life),
- increasing the proportion of extra-mathematical topics,
- connecting mathematical and applicational components more strongly,
- including more real examples and emphasising the processes of translating between the real world and mathematics instead of working with ready-made models only,

- involving students more actively.

In everyday *teaching practice*, the amount of A&M is also increasing. Standard models especially have entered the classroom and are treated to a moderate extent. However, the mainstream of mathematics instruction is still far behind the forefront of research and development in mathematics education. Apart from some model countries or innovative projects, applications, and even more so modelling, still play only a modest role in school and university classrooms. Formal calculations or intra-mathematical considerations are still at the heart of mathematics teaching. Why is the situation like this?

I think this is due to some actual obstacles. These have been well-known for a long time and quoted time and again (see Pollak, 1979, Niss, 1987 or Blum and Niss, 1991), but they still exist. I will summarise four of them.

Obstacles with regard to the **organisation** of instruction. Among other things, the usual school or university organisation interferes with the teaching of authentic, open-ended problems; for some problems – such as the traffic example – the students ought to go outside to make their own observations. The abilities connected with A&M are difficult to assess, and what is not actually and regularly examined will not be taken seriously enough by students or by teachers.

Obstacles from the **learner's** point of view. A&M work makes mathematics learning more demanding and less predictable. Modelling, in particular, requires imagination and creativity as well as solid knowledge of standard topics.

Obstacles from the **teacher's** point of view. A&M work also makes teaching more demanding; among other things, additional time and effort is required to find suitable examples and to get these ready for particular groups of students. Teaching also becomes less predictable, because unusual types of classroom interaction may occur, for example discussions about environmental problems in the case of traffic flow, or evaluation questions such as "Which of the distance models is the best, and why?" Teachers might doubt whether this belongs to maths lessons at all, and might feel their expert authority undermined, which in some cultural environments is a really severe problem.

Obstacles with regard to **materials**. The amount of materials available is not a real obstacle (see part 4 of this paper). However, many existing examples and materials are not integrated into regular curricula, and many have a strong cultural, local or regional touch. In which parts of the world is the case of maximum traffic flow, for example, really relevant? What about economic cases such as taxes, or ecological and social cases such as energy consumption or population growth? On the other hand, taking into account the fundamental global problems we are all facing, as well as the fact that politics and economy on our planet are more and more intervowen (which necessitates global thinking and the awareness that there is only one world) it is even more desirable to treat not only locally-based examples.

4. WHAT MATERIALS ARE AVAILABLE FOR TEACHING MATHEMATICAL MODELLING AND APPLICATIONS? EXAMPLES AND PERSPECTIVES

There is really a wealth of A&M examples and materials for every topic area of school or university mathematics, all in principle accessible to teachers by way of books, project materials or articles in journals. A lot of references can be found in the A&M surveys from the ICME congresses: Pollak (1979), Bell (1983), Niss (1987), Blum and Niss (1991), Blum (1993). In the last few years, many interesting new project materials for secondary schools have been developed, some of them are presented in the last reference above. Here are some additional resources.

A lot of materials has been created in various states of Australia – see for example Finger and Treilibs (1992), and the references in Carr (1993) and in Money (1993). Authentic applications have been prepared for the classroom in Denmark – see the contribution of Ebbensgaard in this volume. In Germany, many detailed teaching units have been developed by the project MUED (Mathematik-Unterrichtseinheiten-Datei), for example Böer (1983). The Freudenthal Institute at Utrecht University (de Lange *et al.*) in the Netherlands has produced lots of materials, including textbooks. In Portugal, there are initiatives aiming at integrating mathematics with real-world situations in the lower secondary level, including project work – see, for example, the description of the MAT 789 project by Abrantes (1993). In the UK, numerous books and pupils' materials have been produced by the CIMT (Centre for Innovation in Mathematics Teaching) at Exeter University (Burghes *et al.*) and by the Shell Centre for Mathematical Education at Nottingham University (Burkhardt *et al.*), both structured according to problems, not to mathematics; the ongoing Nuffield Advanced Mathematics Project (Neill, Burns *et al.*) also contains a strong modelling component including case studies, and the new Welsh project Practical Applications of Mathematics is presented by Tanner and Jones in this volume. Last but not least, several useful books have been produced by USA projects, among others by COMAP (especially the series of HIMAP Modules, 1985-93, for the secondary level and UMAP Modules, 1981-93, for upper secondary and tertiary levels), by UCSMP (the University of Chicago School Mathematics Project, directed by Usiskin), by NCTM, by CORD (the Centre for Occupational Research and Development, 1988-93, with examples from professional and everyday life for the lower secondary level), and by Sloyer *et al.* at the University of Delaware, in collaboration with Tri-Analytics (see, for example, Sacco *et al.* 1987/88).

Many more references to the literature in German, English and French can be found in the extensive bibliography by Kaiser-Messmer, Blum and Schober (1982-1992) and, of course, there are lots of other interesting activities in those and other countries and language areas as well that I could not present here or that I don't know about. I apologise for any omissions.

Nevertheless, the small selection of materials and activities mentioned above demonstrates several promising directions of development:

- case collections for new A&M - oriented mathematics curricula, including assessment, in Australia,
- authentic examples of mathematical modelling in Denmark,
- application-oriented teaching units in addition to conventional mathematics textbooks in Germany,
- application-oriented mathematics curricula on a global scale and suitable textbooks in the Netherlands,
- subject-integrated materials in Portugal,
- examples of activity-oriented global problem sequences in England,
- materials for a new direction of mathematics curricula in the computer age in the USA.

The last-mentioned aspect deserves a special discussion and is detailed below.

5. WHAT IS THE EFFECT OF COMPUTERS IN TEACHING MATHEMATICAL MODELLING AND APPLICATIONS? OPPORTUNITIES AND RISKS

We can clearly see an international trend towards an extended use of computers in mathematics teaching. The opportunities offered by computers are rapidly increasing, and there is indeed no doubt that they can effectively improve mathematics learning and teaching, amongst other things by making it possible to deal with more complex applications, by way of simulation, or by relieving learning and teaching of some tedious activities and thus making it possible to concentrate on higher-level aims such as modelling. One example from our experiences is using DERIVE in the case of traffic flow rate in grade 11 for trying and comparing diverse distance models, varying parameters, drawing qualitative conclusions, analytically calculating the optimum, and so on.

New possibilities like these are well-known and quite often quoted today. What is less often quoted is that the use of computers is accompanied by certain problems and risks, also with respect to A&M. For example, superficial learning may be advanced, and the necessary intellectual efforts of students may be replaced by mere button pressing – see the list in Blum and Niss (1991). These risks are still acute today. Presumably, some have not become visible in public yet because the use of computers in mathematics teaching and its curricular implications are not yet put into effect on a broad enough scale. From our experiences, I can add some more problems.

- The instructional possibilities of computers may be over estimated, for instance concerning difference and differential equations and their applications. Overestimating computers and computer mathematics may result in overly devaluating 'traditional' topics and skills which are still indispensable or, in some cases, perhaps even more important than before. An example is given by the false dichotomy 'continuous vs. discrete mathematics', another is a rash elimination of some basic algorithms which, among other things, are necessary to understand and to judge results delivered by the computer.

- Present software is by no means didactically perfected yet. On the one hand, most professional software is, in a way, too powerful for teaching and learning purposes. On the other hand, most available software requires an excessively great deal of formal effort for solving some standard problems so that, for example, students won't be able to make illuminating discoveries by themselves – such as braking distance equals car length in the case of optimal speed in the traffic problem.

Why do I mention all this? Not in order to prevent, in a small-minded way, anybody from using computers. On the contrary, I am convinced – like many other people – that computers may considerably improve mathematics teaching and that mathematics curricula should be altered today to meet the dynamically changing demands of tomorrow. I mention these frequently neglected problems for two reasons. First, teachers and students should become fully aware of these problems. This, of course, does not solve them, but it will contribute towards a more reasonable use of computers and towards developing meta-knowledge of mathematics and its tools. Second, the problems point to certain open research questions. Some of these – such as the one on the relative importance of elementary algorithms – have actually been relevant for a long time, but now computers bring these questions inexorably into light. What we need in particular are more long-term empirical investigations to explore the actual effects of the use of computers in teaching.

A final remark. I am more convinced than ever that – because of simple logistic difficulties – present day personal computers will not really affect maths teaching in the future but only cheap and permanently available 'pocket computers'.

6. WHAT NEEDS TO BE DONE FOR APPLICATIONS AND MODELLING IN PRACTICE AND IN RESEARCH? MEASURES AND QUESTIONS

There is already a lot being done to overcome those difficulties and to induce changes in the school or university classroom in desirable directions, as indicated in part 3. Therefore we can certainly speak of a trend towards reducing the gap between what is intended in mathematics education and what is achieved in mathematics teaching, though with considerable differences between various countries. Nevertheless, many of those difficulties are, in my view, unavoidable and cannot be 'methodised away' in an easy manner. They have to be taken into account explicitly by teachers and learners and, again, they call for more research and development.

What should be done? I see at least the following practical measures.

1) Developing appropriate **modes of assessment** for A&M. 'Appropriate' means, generally speaking, that such modes have both to reflect the nature and spirit of A&M and to comply with the usual needs of formalised testing. For most educational systems, these are conflicting demands. I am not sure whether it is possible at all to assess all important aspects in a reasonable way, nor am I sure whether this is desirable. Concerning these and other fundamental questions of assessing mathematical A&M, I refer to Niss (1993c).

   During the last few years, the topic of assessment has become more and more important in mathematics education in many countries. We can even see a trend towards conceiving new modes of assessment, especially with respect to A&M (see also Niss, 1992). Many very interesting contributions, both to theory and to practice, are contained in the two ICMI studies Niss (1993a,b). For the topic of A&M, the materials and experiences (both good and bad) from the Netherlands, from Australia, from England and from Denmark seem to be particularly valuable – for example see de Lange (1993), Money and Stephens (1993) as well as the contributions of Galbraith and of Haines and Izard in this volume.

2) Initiating all kinds of in-service and pre-service **teacher education** activities in order to supply as many teachers as possible with knowledge, abilities and, in particular, with attitudes to cope with the demands of teaching A&M. All aims that students are to achieve have to be achieved by teachers themselves. Especially important, in my view, are local or regional in-service courses where teachers teach teachers.

3) Encouraging teachers to **organise themselves**. One very promising example is the German group MUED mentioned in part 4. Founded in 1977, it consists at present of more than 500 members, mostly teachers from all types of school. A description of the philosophy of MUED is given by Böer and Meyer-Lerch (1989). The group has developed up to now more than 500 teaching units, several of which are available from a commercial publisher. This third point is one facet of a general and vitally important issue: recognising and advancing teachers' professionalism.

4) **Developing new A&M materials** which are or can easily be embedded in regular curricula and which meet certain requirements of educational quality, and developing new textbooks and curricula with A&M as an essential component. This has to be done in each country separately, but of course profit can be gained from materials and experiences from other countries, too, especially since there are common principles of curriculum construction (compare Usiskin, 1989 and 1991), one important being the need for structure.

I have called these measures practical. However, several of these belong to **research** as well. For, in my view, the term research also includes developmental work guided by theory or systematic reflections upon fundamental issues. I have addressed research questions in various other parts of this paper, too. I think it is time and highly desirable to intensify research activities in the field of A&M in learning and teaching mathematics (see also Ponte, 1994). I will close by giving a list of possible research aspects – a research programme. I do not claim that these aspects are new or complete nor that there have not already been many contributions to individual aspects. What I would like is for research work related to aspects neglected so far to be intensified and, in particular, for a coherent view of these multifarious aspects to be considered in all individual research activities. Almost all the following questions have to be answered in relation to specific educational situations (age, ability level and so on).

Fundamental and Curricular Aspects

- What are the aims of A&M in maths teaching, and what is their relative importance? How is A&M interrelated with other aspects of maths education?

- What are suitable cognitive models related to learning mathematics in applicational contexts, and what should the essential abilities of students be?

- What are suitable concepts and cognitions related to teaching mathematics in applicational contexts, and what should the essential abilities of teachers be?

- What are appropriate kinds of examples? How can these be prepared for the classroom? And: Developing examples, teaching units and projects for certain purposes.

- What are appropriate conceptions for combining mathematical and applicational components in the curriculum? And: Developing curricula for certain purposes.

- What are appropriate modes of assessing A&M skills and abilities? And: Developing assessment schemes for certain purposes.

- What are appropriate modes of evaluating A&M materials and programmes? And: Developing evaluation schemes for certain purposes.

- What are appropriate media for teaching A&M, how can they be used, what are their implications? And: Developing ideas for using media (especially computers) for certain purposes.

Empirical Aspects

- What are individual students actually capable of?
  Assessing student's attitudes, skills, abilities and difficulties in connection with certain A&M materials or programmes (including media) and in relation to certain learning models, and comparing different individuals or groups, also on an international level (in small-scale case studies or large-scale statistical investigations).

- What are individual teachers actually capable of?
  Assessing teachers' attitudes, capabilities and difficulties in A&M contexts.

- What actually goes on in the classroom?
  Observing and analysing teaching, learning, communication and interaction processes in lessons with A&M work.

- What are the actual effects of given materials?
  Evaluating and comparing A&M materials, programmes and

assessment schemes.

Epistemological and Philosophical Aspects

- What could 'real world' mean, and what are (models for) its relations to 'mathematics'?
  What is a 'real situation' or a 'real problem'? What could terms such as 'application', 'model' or 'modelling' mean in such a framework?
  In particular, is it possible to find answers to these questions by theories borrowed from other sciences?
  How have these concepts developed in the past?

- What kinds of models exist?
  How does the status of a model depend on the mathematical topic areas and the extra-mathematical fields related to it?
  What was and what is the social use of models and modelling?

- What are the basic philosophical positions of various conceptions for application-oriented mathematics instruction and education?
  How have these conceptions developed in the past?

All these questions are related to A&M. However, they cannot be answered without referring to more general issues. For instance, the question of aims and conceptions for A&M depends on the general aims of education and especially on the underlying conception of human nature. Determining students' abilities or difficulties with A&M is not possible without using (unconsciously perhaps) some theory of learning. Asking for the concepts of 'application' and 'modelling' depends heavily on a conception of 'mathematics'. More than that, these general issues, too, are interrelated (as usual in pedagogy and education – see Freudenthal, 1983). They all rest (to adapt Thom, 1973), whether one wishes it or not, on a philosophy of mathematics education. The extent to which such fundamental issues are developed is an essential indicator of how far mathematics education, the didactics of mathematics, has progressed as a scientific discipline.

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2

Modelling, Teaching, Reflecting – What I Have Learned

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1. INTRODUCTION

Interest in the teaching of mathematical modelling and applications had its genesis through the efforts of innovative individuals and groups, motivated by beliefs about the importance of doing life-related mathematics. A development can be discerned, over a period of a decade or more, that has broadened the scope of modelling and applications from an essentially classroom-based interest to one that extends beyond content and learning situations to involve also conceptions of institutional learning, social implications, and political issues. This broadening scope is exemplified within the contents published as proceedings from ICTMA-5 (de Lange, Keitel, Huntley, Niss 1993). In this volume, for example, de Lange discusses modelling in relation to changing goals of Mathematics Education, Keitel relates applications to environmental issues, social responsibilities, and interdisciplinary learning, Julie indicates how modelling can promote an emancipatory role for mathematics through addressing social problems with political content, and Niss addresses the tensions generated when official assessment requirements compromise the integrity of creative initiatives in the teaching of modelling and applications. Ormell reminds us that mathematical modelling is a 'priceless discipline' and, in reviewing the relevance of a variety of pedagogical approaches, emphasizes again the need for a higher synthesis promoting not only enhanced classroom expertise but learners empowered for life.

In parallel with these broadening emphases evident among the ICTMA papers, the System Dynamics community has been engaged in developments to introduce systems thinking and modelling to curricula at all levels (Richmond, 1993; Draper, 1993; Davidsen, Bjurklo and Wikström, 1993). These initiatives also aim at the development of modelling pedagogy (causal loop modelling), learner empowerment, and the use of models and the modelling process to address issues with social and political content.

It can be noted then, that reflecting upon the teaching of mathematical modelling and applications requires that the reflection includes, but is not limited to, events in tertiary classrooms and schools.

More specifically Blum and Niss (1991) in a state of the art review, listed five arguments on behalf of mathematics that have formed the basis for including applications, and more recently modelling, in curricula. These are respectively:

(a) Formative argument – the role of modelling and applications in developing general competencies and attitudes in students.

(b) Critical competence argument – preparing students to be perceptive and competent members of an increasingly mathematized society.

(c) Utility argument – preparing students to use mathematics in problem solving, on the assumption that this ability does not necessarily follow from training in pure mathematics.

(d) Picture of mathematics argument – applications and modelling are needed to round out a proper cultural experience in mathematics.

(e) Promoting mathematics learning argument – applications and modelling act as motivating influences for mathematical study, and contribute to the content of relevant mathematics learning.

Whether on the basis of one or all of these arguments the challenge to incorporate applications and modelling in the mathematical experience of students retains pre-eminence as an issue from primary through tertiary education.

2. CONTEXTUALIZING MODELLING IN EDUCATION

It is taken as axiomatic that a modelling problem can be addressed at various levels. For example, modelling situations can be highly structured, or presented as problems to be formulated and resolved in a totally open manner. In the former case the problem formulation is substantially provided, and emphasis in student work focuses on solution and interpretation. In the latter case a full range of modelling skills is required, in particular the ability to formulate an initial model. In order to develop confidence and competence, and for applications and modelling to fulfill purposes such as those listed above, it is argued that students need experience across the range of emphases. It is by no means clear, however, that there is sufficient agreement among modelling practitioners to convey unambiguous messages to those taking first tentative steps. If the gap between the perceived requirements of modelling and existing practice is too great, potential innovators find the prospect too daunting. It seems essential to provide graded support from a familiar base.

One classification that has proved useful in assisting teachers to develop modelling expertise from a starting point of standard applications is illustrated below. Three levels of "modelling" are identified

A generalized applications
B structured modelling
C open modelling

3. GENERALIZED APPLICATIONS

The starting point is a standard application of the type found in large numbers in conventional text materials.

Example

A person in a boat B is 3km from the nearest point 0 of a straight beach. The destination D is 6 km along the beach from 0.

(i) If (s)he can row at 4 km/h and walk at 5 km/h, towards what point on the beach should (s)he row to reach the destination in the least time?

(ii) Solve the problem if the rowing speed is changed to $4\frac{1}{2}$ km/h.

This is a standard application of differential calculus (see Fig. 1). Suppose the person rows in a straight line from B to C, and then walks from C to D. Let the point C be x kilometers from O where $0 \leq x \leq 6$.

(i) Denoting the total time of travel by $T(x)$ hours gives

$$T(x) = \frac{\sqrt{x^2+9}}{4} + \frac{6-x}{5},\ 0 \leq x \leq 6$$

$T'(x) = 0$ gives $x = 4$ for the minimum time.

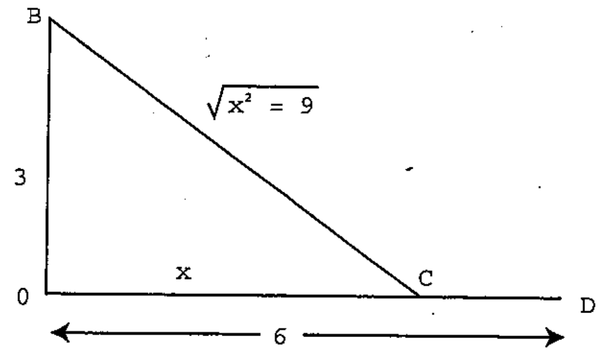

Fig. 1

(ii) $T(x) = \frac{2}{9}\sqrt{x^2+9} + \frac{6-x}{5}, 0 \leq x \leq 6$

$T'(x) = 0$ gives $x = 6.2$ which is beyond the domain of T. Since T defines a decreasing function the minimum occurs at the endpoint of the domain, i.e. $x = 6$. The person should row directly to D, and this is the end of the solution to the problem as set.

Introducing a Modelling Aspect

A slight change in rowing speed (from 4 km/h to $4\frac{1}{2}$ km/h) has altered the solution dramatically. Mathematically the change is from a stationary point minimum to an end point minimum-physically from a rowing plus walking solution to a rowing only solution.

Viewing these as two special cases of a more general situation enables the modelling of a higher level problem, i.e. to find when a solution will fall into one or the other of the above classes. Since the change in the nature of the solution was produced by a change in speed, we shall allow the speeds to be the variable parameters, r and w, say. Then the time function is given by

$$T(x) = \frac{\sqrt{x^2+9}}{r} + \frac{6-x}{w}, \; 0 \leq x \leq 6$$

$$T'(x) = 0 \text{ then gives } x^2 = \frac{9r^2}{w^2 - r^2}$$

For a rowing plus walking solution we need $0 \leq x \leq 6$

It is easily checked that this condition is satisfied for the speeds in part (i) of the application but not for the speeds in part (ii). Further generalizations are possible, e.g. replacing the distances OB and OD by generalized values, a and b, enables the upper bound in the above inequality to be expressed in parametric terms.

Thus the problem can be explored, not just in terms of particular calculations, but as an investigation of factors that determine the nature of the outcome – a generalized model. The initial data have been viewed, not as ends in themselves, but as special cases underlying a more general principle.

Insights for Modelling

One person viewing the original question may see only a single application achieved by manipulating specific arithmetic values. However, a modeller conceptualizes a whole family of solutions of which the given context provides but one. In generalizing given constants to variable parameters the modelling potential in a situation can be developed. In following this path it is necessary to make decisions such as the following.

- Which constants to "parametrize"?
- What physical limits to place on the values that can be assigned to the parameters?
- What mathematical results follow from the generalized approach?
- How are these results interpreted in terms of decision making for the contexts being explored?
- What further generalizations are useful?

Answering these questions involves choosing variables, making assumptions, solving mathematics, interpreting results and evaluating implications – all recognisable components of mathematical modelling.

In the sense that we began from a standard application, a modelling pedagogy can be derived for such situations – based initially on the 'parameterization' of given constants. To the extent that this approach has an identifiable starting point and a common structure it can prove helpful in the development of modelling skills – as a learnable and teachable approach. Further, there is no shortage of examples from which to begin.

4. STRUCTURED MODELLING

Here general contexts found in reality are considered. However to assist the students, organising statements are provided to chart them through essential aspects of the process.

Example

Explain the design of the cardboard core used in toilet rolls and paper towels. Design a core whose circumference and height are to be twice those of your own sample.

A structured modelling approach might be provided as follows – the diagram shows the plan of an opened-out core (see Fig. 2).

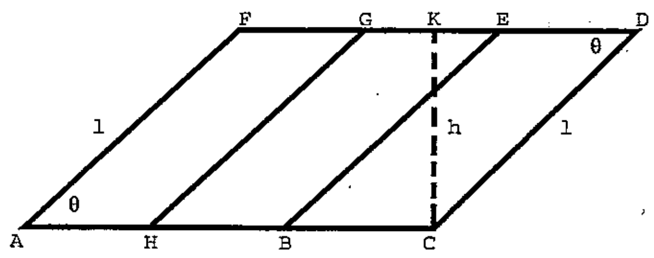

Fig. 2

(1) Unwind the roll to form a plane figure – identify and measure equal edges.

(2) Find lengths that are equal to the circumference of the roll.

(3) Calculate the area of the whole parallelogram (AFDC).

(4) Find other values for l and θ that will form a core of the same radius and height.

(5) What are the smallest and largest possible values for θ? What guides the choice of θ?

(6) Design a core whose circumference and height are to be twice those of your sample.

Approach to Solution

The core consists of two congruent parallelograms, $ABEF$ and $GHCD$ gummed together to overlap such that $HBEG$ is a double thickness and $BCDE$, $AHGF$ are single thicknesses. $AB = 2\pi r$ and $BC = \pi r$ where r, h are respectively the radius and height of the core. $BCDE$ wraps onto $AHGF$ forming a cylinder of double strength – it can be verified that CD and HG coincide to form an equi-angular spiral on the surface of the cylinder. All lines (like CK) that are parallel to the vertical height cut these edges at a constant angle. Given r and h and the foot of the perpendicular K, the parallelogram is constructed by calculating l and θ where

$$l^2 = DK^2 + KC^2 = (2\pi r - KG)^2 + h^2 \text{ and } h = l \sin\theta$$

For an actual core with $r = 2, h = 10, GK = 1.5$ we get $l = 17.25$ and $\theta = 35°$ (approx).

Using the constant area locus a variety of values of l and θ can be found for a given radius and perpendicular height. These correspond to different positions of K. Maximum ($\theta = 90°$) and minimum ($\tan\theta = \frac{h}{3\pi r}$) values for θ can be found and related to the design and strength of the core. Requiring the design of a core with twice the dimensions is one way of assessing that the mathematical principles have been understood and can be applied. When a modelling situation is structured for students, the application of the inherent principles to a variation of the problem may be an essential element in assessing modelling competence.

5. OPEN MODELLING

Here students are asked to make progress with problems based on real situations without assistance – to carve mathematics from the rock face.

Example

The State Government is considering building a controversial major road through the northern and inner suburbs of Brisbane.

Resident action groups have been very vocal in this opposition, claiming it will have a major effect on their lifestyles. One of the options being considered is the building of a tunnel under one of the hills in Bardon, rather than above ground, thereby causing less disruption to residents in the area. Owing to financial restrictions, the tunnel would probably have one lane in either direction. The Chief Engineer has realized that there will be holdups at both ends of the tunnel during the morning and evening rush hours.

Bearing in mind aspects of safety, and the desire to produce the maximum flow of traffic at peak times, she wishes to put up signs indicating a maximum speed and the distance to be maintained between vehicles. You have been contracted to work on this project. What recommendations would you make to the Chief Engineer?

For this type of problem a systematic and incisive approach to modelling is required. A full range of skills from problem formulation to solution validation or refinement needs to be developed. A discussion of this problem is included in Clatworthy and Galbraith (1991).

Type A modelling is located near the application pole. It contains many elements of creative modelling and can be sourced in terms of a variety of applications available in curriculum materials. In basing this activity on generalizations of existing applications, a variety of content areas and levels can be incorporated. This type of modelling approach can be based around generalizable mathematical and pedagogical strategies, and assessment may be possible using variations of standard practices. Modelling activity of this type does not address the issue of formulating an initial model from a real situation.

Type B modelling may be more properly described as a form of structured investigation. Both the context and model data (measurements, etc.) are grounded in real situations and students are led through stages that are designed to display (and teach) various phases of modelling activity. It is not, however, obvious that this alone would suffice to enable students to become independent - in particular the formulation phase of modelling is provided. The level of mathematics incorporated in a model will be a function of the structuring, and hence teacher controlled.

Type C modelling is the only one (of these versions) that requires students to systematically develop skills of formulating mathematical models from complex realities. Consequently it is an approach that must be incorporated into a programme if the aim is to enable students to unlock their mathematics for substantive use, as well as for satisfying assessment requirements.

The three approaches to modelling are intended to be generic but not definitive. They are certainly not mutually exclusive, as approach A, for example, can be used to extend a model initially developed using approach C. Indeed aspects of A and B can be used to assist students to deepen models built initially using an open approach. The refinement stage of a basic model gives an opening to provide structured suggestions on how the model might be improved, without interfering with the basic formulation that will have been accomplished within the basic model. Each modelling approach has been found to make a contribution, both with respect to mathematical achievement and with regard to the logistics and constraints of teaching situations.

6. THE IMPORTANCE OF INFRASTRUCTURE

Mathematical knowledge is important. As Simon (1980) pointed out "research on cognitive skills has taught us ... that there is no such thing as expertness without knowledge". It has been claimed at various times and places, that the mathematical skills students can call on for modelling purposes, lag substantially (2 years or more) behind their pure mathematical expertise. Experience from at least one modelling programme, Galbraith and Clatworthy (1990), suggests that this observation needs qualification. It was certainly true at the outset of the programme when students entering their senior years at a secondary college could think no further than basic arithmetic for use in their first attempts at models. However, by the end of the two year course they were invoking, and successfully using, new mathematics such as calculus and computer applications that had been learned only weeks or months before. Furthermore, they were using the new content in the solution of unstructured modelling (type C) problems where the formulation was their total responsibility. What brought about this changed capacity? Arguably it can most likely be attributed to the development of what might be called *modelling infrastructure*. This *infrastructure* involves the development of a systematic, strategic, approach to the development, interpretation, and testing of models.

An early and widely known support for such development became known as the Open University seven-box diagram that originated some time ago through instructional materials developed by the Open University (UK). Many variants of this diagram exist, such as Fig. 3, and others have devised their own preferred schemes. Fig. 4 contains partially developed infrastructure representation from the more specialized field of System Dynamics modelling.

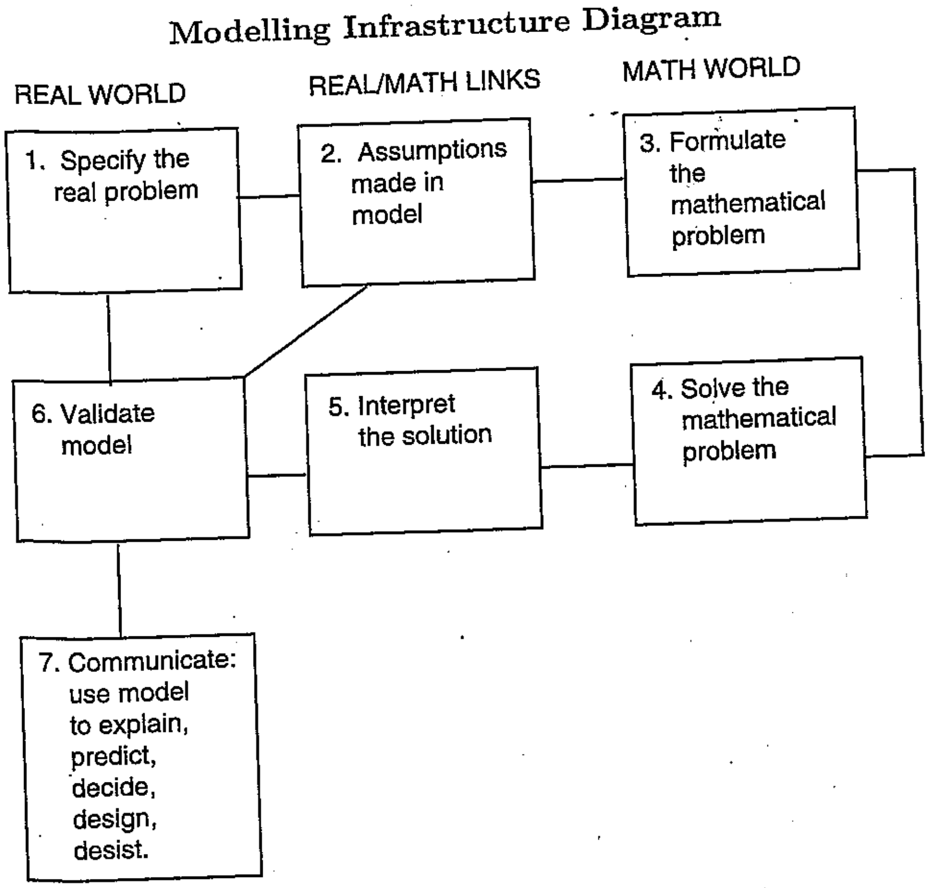

Fig. 3 (After: Open University, 1978)

Other Critical Thinking Skills	Dynamic Thinking Closed Loop Thinking Generic Thinking Structural Thinking Operational Thinking Continuum Thinking Scientific Thinking

Fig. 4: Critical Thinking Skills (after Richmond, 1993)

The significant point is not which *schema* we favour, but that a suitable one exists and can be communicated and used by learners. Evidence has accrued, both formally sought, circumstantial, and anecdotal, that such articulated schemes foster abilities essential for effective modelling. In one programme, Clatworthy and Galbraith (1991), it was not just that students reported the modelling diagram as the most important facilitating factor more often than any other single aspect of mathematics or pedagogy within the modelling programme. It was also the way it provided for transfer of their modelling approach to applications outside the mathematics subject, and outside the college itself. For example one student elaborated carefully on his new approach to structuring solutions to physics problems, while another described an application to a hobby involving the growing of tomatoes hydroponically. Reflecting on experiences and outcomes indicates that the explicit provision of such a structure is an essential aid to the development of modelling skills. So while the development of such schemas may have been originally motivated by pragmatism (it seems to help), we should now recognise their place as a theoretical construct for the learning of modelling.

The importance of such structuring in the successful teaching and learning of modelling strategies receives independent support from cognitive psychology.

Anderson (1990) has written extensively on the development of expertise. In terms of modelling and applications his following points are of interest (pp 256-257).

(1) *Tactical learning* (accomplishing a particular goal) refers to the improvement that comes about because people learn familiar subsequences of problem solving steps that appear in multiple problems.

(2) *Strategic learning* refers to improvement that comes about because people learn the optimal way to organise their problem solving for a particular domain.

(3) *Proceduralization* refers to the process by which people convert their declarative, factual knowledge of a domain, into a more efficient procedural representation.

(4) *Problem solving* improves in a domain because people learn how to represent problems in the domain in terms of abstract (not surface-level) features that facilitate the problem solving.

These points refer to generic aspects of expert performance of the kind facilitated by structures such as Figs. 3 and 4. Richmond (1993) has referred specifically to the importance of explicit appreciation of the modelling process, for workshop participants learning system dynamics modelling.

> "Specifically: (1) tell people that they're going to be asked to juggle multiple thinking tracks simultaneously; (2) be explicit about what these tracks are; and (3) align the curricular

progression to emphasize development of only one thinking skill at a time."

The common achievement in all representations is the reduction of cognitive load, and this seems absolutely essential to the development of successful modellers. As experience increases, the use of generic forms can be used to further reduce cognitive load within sectors of the modelling process. Further the modelling process itself becomes automated. This becomes evident when students who initially make heavy use of a modelling diagram begin to move smoothly between phases in the modelling process without overt recourse to any form of procedural representation. A successful modelling infra-structure representation is the author of its own demise.

To summarise the significance of modelling infra-structure we might use a metaphor borrowed from Vygotskian learning theory. Students who have never undertaken modelling of real life situations have an actual development level which is very basic as far as modelling is concerned, even if they have well developed pure mathematical knowledge. This advanced knowledge can support a much higher level of modelling than they typically exhibit.

The provision of modelling infra-structure through leadership from a competent teacher using relevant pedagogy enables learning in the zone of proximal development to take place. An increased capacity to utilize mathematical knowledge is the outcome of this learning, a capacity which had previously been inhibited by the absence of the infra-structure, rather than by lack of ability or of pure mathematical knowledge.

Apart from its role as a meta-cognitive aid, an infrastructure diagram helps to emphasize concepts about modelling that are important in orienting beginners. One such concept is that modelling is not a subset of mathematics, but extends across discipline boundaries and into the real world. Another is the holistic nature of modelling, with its total quality formed through inter-relationships between respective phases. Thus a model utilizing simplified mathematics may represent a more valid approach to a problem than a sophisticated attempt that leaves parts of the modelling process inadequately addressed.

7. DEVELOPING AUXILIARY SKILLS

Modelling infra-structure refers to an organisational framework that provides essential cognitive and meta-cognitive support for the development of modelling expertise. Auxiliary skills refer to abilities which enhance the capacity of students to learn effectively. They include skills of effective group learning, oral reporting, and project writing.

Small group work features in many programmes involving the application of mathematics to real problems. Approaches to group learning and learning through social interaction are much influenced by the writings of Vygotsky (1978) and of Wertsch, Minick, and Arns (1985). Essentially the relevance for present purposes rests upon Vygotsky's theory that

(1) Learning promotes development

(2) The way this occurs is explained by the concept of a *zone of proximal development* (*ZPD*), where

> "*ZPD* is the distance between the actual developmental level as determined by independent problem solving, and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers" (Vygotsky, 1978 : 86)

Thus the *ZPD* defines those functions that are in the process of maturing rather than those that have already matured; "they are the buds rather than the fruits of development."

(3) Fundamental to learning is the transformation of an inter-personal process into an intra-personal one.

> "Every function in the child's cultural development appears twice: first on the social level, and later, on the individual level: first between people, and then inside the child."

> "Learning awakens a variety of internal development processes that are able to operate only when the child is interacting with people in his environment, and in co-operation with his peers."

However, while effective group learning makes a major contribution to the development of modelling expertise, the mere presence of group activity does not guarantee quality learning.

Tudge (1990) investigated group learning in terms of Vygotskian theory and inferred (from work with younger children) that peer collaboration can have a powerful impact that is not always positive. He distinguishes between interaction containing an acknowledged more competent member (e.g. teacher or recognised peer) and peer interaction where the authority of knowledge is less evident and accepted. Tudge found that a higher level of thinking was likely to improve partner learning if supported by a confident manner and convincing reasoning. However an inferior rule, confidently held, was found to induce regression among students, whose understanding, though at a higher level, was tentatively held. He inferred that immediate feedback is important, in the form of evidence external to group opinion against which group members can test attempted explanations or solutions. A similar phenomenon was reported by Goos (1993) in a study of senior college students collaborating to solve unrehearsed problems in mechanics. Once again inferior strategies, confidently argued, diverted students from more productive avenues tentatively suggested.

This research alerts us to the proposition advanced by Tudge that a ZPD does not just extend forward from a learner's present state; rather it surrounds him or her, and regression as well as progression is possible. Clearly group composition and the position and status of members is of potential significance.

Hatano and Inagaki (1992) claim that constructive interaction occurs frequently only in certain types of groups. Such groups are characterised by the following aspects.

(1) *Being horizontal in terms of flow of information* – the key here is that challenge and elaboration occurs between perceived equals i.e. perceived expertise is changeable.

(2) *Containing three or more members* – there is value in the presence of an audience beyond two members

   (a) socially – since debate will be livelier to seek support of other parties,

   (b) cognitively – since third parties give clues for evaluating the worth of arguments between proponents and opponents.

(3) *Involving empirical confirmation* – interaction is induced when group members debate alternatives which are falsifiable by empirical means. Such occurs when group activity is 'situated' in a specific context.

   For the purposes of this paper the construction/evaluation of a mathematical model would provide such a context.

(4) *Room for individual knowledge acquisition* – group problem solving may not result in all members acquiring its product i.e. what has been achieved collectively may not be coordinated into a new piece of knowledge in individual heads unless it is explicitly articulated and represented.

   In a mathematical modelling programme the requirement that group members provide individual reports can be used to address this issue.

In summary Hatano and Inagaki conclude that two processes are involved in the effective construction of knowledge through group interaction:

(A) the individual invention of knowledge stimulated by group motivation,

(B) the assimilation of information proposed by others during interaction, modified by individual editing.

Findings such as the above suggest that the form of the group process used to enhance capabilities in mathematical modelling is important, and needs to be carefully organised and monitored.

We have found it useful to reflect upon the form of group process used in our programme, in view of the above research of which we were unaware at the time.

(1) Groups consisted of 3 or (at most) 4 individuals.

(2) Group composition was changed for each case study.

(3) A group training exercise was provided to teach cooperative problem solving.

(4) Groups, once formed, were autonomous and scheduled their own meetings in addition to those provided for within normal timetabling. These were arranged at times and places decided by group consensus.

(5) Individual members of groups were interviewed separately on the subject of their case studies.

(6) Each individual member was responsible for their own project report.

(7) Each group made a video presentation to the class.

In retrospect we recognise that our emphasis was on the individual acquisition of knowledge facilitated by group processes. Group learning as such was not assessed. The small group activity was supplemented by whole group plenary sessions. These were interspersed while the class was working on the training problems. Plenary sessions provided the venue for the sharing and debating of ideas generated within the groups and were conducted under the guidance of a student chairperson. The teacher's role was one of monitor, guide, judicious curtailer of wild goose chases, and subtle encourager of potentially fruitful suggestions and directions for exploration. As a participant the teacher filled the role of 'more competent other' in the Vygotskian sense, and endeavoured to channel the social dialogue in productive ways.

Other auxiliary skills involved the training of students in oral reporting and project report writing. In both these areas opportunities were taken to involve staff from other disciplinary areas (English and Drama). The extent to which effective articulation and communication enhances learning (Rieber and Carton, 1987), makes investment in the development of these skills an important supplement to the modelling and group processes comprising other aspects of the pedagogy. Furthermore it helps to emphasize that modelling involves interdisciplinary collaboration — all requisite wisdom is unlikely to reside within any single head.

8. ASSESSMENT

It scarcely seems necessary to argue that mathematical modelling as a constructive activity is supported by a constructivist epistemology. Few would insist that a model can be "given" to someone as a unique description of an objective reality. Within mathematics education writers such as von Glasersfeld (1987) and Wheatley (1991) have argued the constructivist case. Schoenfeld (1985, 1987) has written extensively from this perspective within the field of mathematical problem solving.

Modelling pedagogy, insofar as it encourages discussion, debate, and active construction and defence of problem representations (which involves self-regulatory meta-cognitive activity), is deeply embedded in contextual settings. Consequently the means whereby such capabilities are assessed should derive from the same base. Richmond (1993) describes approaches to the measurement of educational performance under conventional teacher-directed learning.

> "Simply ask the student to re-transmit what has previously been transmitted by the teacher. If the student can "dump" a full load he is performing well."

Richmond goes on to note how the conventional classroom does not assume that students have much to contribute to each others' learning.

> "Otherwise they would not be arrayed in a physical arrangement in which they face the back of each others' heads."

Clearly the form of assessment attached to this learning context is inappropriate to evaluate the quality of modelling and application work.

Niss (1993) draws a nice distinction between emphases in the assessment process. If, he argues, the prime purpose is assessment, and modelling and applications merely constitute a domain, then the needs of assessment will dominate the choice of applications and models. If, on the other hand, interest is primarily on the development of modelling abilities, then assessment must be made to serve this end.

It is difficult to reconcile the concept of a standardized test, or even a standard problem, for the purpose of assessing a student's ability in modelling. Proponents would appear prepared to sacrifice validity for some spurious notion of reliability. It is argued that a wrong assessment "model" underlies such thinking. Rather than basing assessment methods around the concept of a common test, the "model" for creative mathematical activity is more properly the research dissertation, that is evaluated qualitatively in terms of criteria and standards that are embedded in the profession. This is particularly so when a major goal is to cultivate the ability of students to model problems of relevance to their own situations and which are, in consequence, particular to certain contexts and interests. No-one believes that higher degree candidates should be assessed against each other by requiring them to address the same thesis topic!

Clearly much remains to be done to develop agreed criteria and standards that can be consistently applied at a system level. The impact of nationally accredited assessment procedures cannot be ignored. However, accepting the imposition of positivistically based assessment procedures, upon mathematical activities grounded in constructivism, represents a philosophical contradiction that can never achieve beyond second best. An abiding challenge, and one being addressed in several countries, is the establishment, communication and application of qualitatively based assessment criteria as measures of modelling quality. To the extent that the teaching community is larger and less homogeneous than the mathematical academic community, this task of embedding standards within the profession is commensurably greater. It must, however, be pursued relentlessly.

9. BEYOND MATHEMATICS AND PEDAGOGY

Reading accounts of classroom experiments, sharing experiences with others, and reflecting on one's own experiences, results consistently in sensations of *deja vu.* Similar issues, experiences, outcomes, and anecdotes appear in many different contexts within modelling programmes. Some of the most vividly reported include

(i) the exceptional quality of work performed by some students,

(ii) the degree of intensity and motivation demonstrated by students when immersed in modelling problems,

(iii) disbelief among colleagues that students could have produced, unaided, work of the quality achieved,

(iv) statements from at least some students that modelling had changed their view of mathematics, if not their life.

If anything can be said to characterise "true believers", perhaps it is that we have each experienced absolute awe at what can be achieved when the motivation and creative power of students is unleashed. An inevitable product of such realization is that certain conventions associated with mathematics and education are challenged, at least privately.

Some of these conventions are

(a) conventions about what constitutes appropriate and realistic mathematics at school, college, or university,

(b) conventions of pedagogy,

(c) conventions of assessment,

(d) conventions about mathematics as a social agent,

(e) conventional expectations of student behavior and capacity to learn,

(f) conventions associated with institutionalized learning.

The implications of these realizations may be treated piecemeal, or we may seek to embed them coherently in a broader theoretical framework such as provided by constructs such as *structuralism* and its radical offspring *post-structuralism* (Gibson, 1984 and Norris, 1982). In summary structuralism postulates that certain basic structures govern and explain any object of study. Central assumptions of structuralism include that reality is expressed, not through individual structural units, but in the *relationships* between them. Structures survive through the operation of internal rules and transformations that also constitute the origin and direction of change. One consequence is that individuals tend to be marginalised with their actions determined by the structure in which they are embedded.

Of particular relevance to the present discussion are relationships that exist between key words and the objects for which they stand. At a general level this is representative of links between *signifiers* and that which is *signified*. Structuralism holds that the relationship between a sign and that for which it stands is arbitrary – only convention attaches meaning within a given structure.

In our domain, key terms include "mathematics", "classroom", "teacher", "school", "assessment", etc., and within our systems of education well-defined relationships have meant that meanings attached to such terms are anything but arbitrary. They have been given meanings through the agency of the systems to which they belong, and the individual who wishes to be different is likely to feel the full impact of structural coercion.

Post-structuralism seizes on the notion of arbitrariness between a word and its meaning to deconstruct the idea of structural stability. Pushed to its extreme, this argument denies the possibility of any meaning at all. At a less extreme level it asserts the legitimacy of relationships other than those holding the dominant position within a given structure. Critical theorists use this approach to challenge aspects of the social and economic order that they believe are inequitable and disempowering to the vulnerable. In education, this argument holds that existing frameworks are used to control and socialize students and teachers into accepting certain givens that operate to the advantage of those in positions of power. As indicated above, there are structural forms that have come to define what it means to teach, how students in classrooms should learn, how schools and other institutions should operate, how mathematics should be presented, and what counts as credible assessment. Let the individual dare to differ from accepted norms!

In the teaching and learning of mathematical modelling we have a vehicle that challenges much of this accepted structure. The concept of modelling itself challenges traditional views concerning mathematics teaching, within both the mathematical and the educational communities. But beyond this, the approach taken within modelling and applications can challenge far more fundamental relationships. Modelling examples that address issues of environmental and social concern, or that specifically demonstrate disadvantage and inequality, convey a different message from examples restricted to problems advancing the cause of consumerism. Mathematics can become a weapon for public and political persuasion. Further, a challenge to accepted structure comes through the agency of modelling pedagogy. In stepping down from the podium, in sharing uncertainty with students, in adopting a facilitating and enquiring role, rather than an authoritative one, the concept of the typical mathematical pedagogue is challenged. In taking control of their own learning, in initiating classroom learning episodes, in working intensively without close direction, students challenge the stereotype of what a person learning mathematics is supposed to do. In meeting at unusual times, in leaving classes and school premises as required to gather data, the meaning of school as a centre of institutionalized learning is challenged.

In summary it is argued that collective experiences, reported from case studies, indicate that the teaching of mathematical modelling and applications provides a systematic and sustained challenge to conventional interpretations of mathematics teaching, learning and assessment. As such, while existing as an entity in its own right, mathematical modelling can be viewed as a coherent part of a wider movement for educational and social renewal. It can legitimately be represented as a transforming political and social enterprise.

10. MATHEMATICAL MODELLING AND GROUNDED THEORY

It would be fair to assume that almost all those working in the area of modelling and applications have been trained in the traditional scientific paradigm. Hypothesis testing through controlled experiment is the familiar method of advancing knowledge. A substantial amount of experience, data, and perceptions has accrued from many case studies in the teaching of modelling and applications. The need has been expressed for the development of further theoretical perspectives. As has been noted previously, modelling is a constructive activity of considerable complexity. Decisive theoretical advances will not occur through attempts to test hypotheses based on positivist assumptions involving dubious controls and spurious outcome measures chosen for measurement convenience. To attempt to do so is epistemologically unsound.

The challenge is to synthesize existing and future information into a theory to further enhance practice. Needed is a method of generating and testing theory from data that is consistent with constructivist principles that underpin its purpose and methods. Such an approach might be developed through the Grounded Theory procedures developed originally by Glaser and Strauss (1968) and subsequently elaborated in works such as Strauss and Corbin (1990). This approach derives from a phenomenological stance, and is theory generating rather than designed to test pre-existing hypotheses. Every type of data is useful; formal and informal, quantitative and qualitative. Such theory is grounded and tested in practice, and may evolve either as a codified set of propositions or as a running theoretical discussion. Above all it is derived from working data, and knowledge gained from naturalistic settings, not from contrived experiments. As a possible vehicle for theoretical advancement, its potential deserves to be explored as we seek to take the teaching of modelling and applications into a new phase of development.

11. SUMMARY

So what observations result from reflecting upon developments in the teaching of modelling and applications? Clearly, if perhaps tritely, what has been learned is that questions and needs continue to emerge along many fronts.

There is a continuing need for case studies at all levels, from primary to tertiary education. Desirably those yet to come will build upon the knowledge and experience so far gained, thus keeping to a reasonable minimum the re-invention of wheels. Of particular importance will be the dissemination of information about system wide experiments that confront issues not experienced in programmes located within particular institutions.

The persistence of meta-cognitive aids such as the Open University modelling diagram and its various offspring attests to the importance of providing specific modelling infrastructure support to learners within teaching programmes. Links with research findings from cognitive psychology, as well as evidence from field testing, provide the potential for further and more precise articulation. We need to move beyond the pragmatic "do it because it seems to work" to a substantive theory for the development of modelling expertise.

Much also remains to be learned about ways to make group learning more effective and accountable. Again, a wider literature exists into which the modelling community could well reach for assistance to meet this purpose. Vygotskian and neo-Vygotskian learning theory appear highly relevant to an endeavor in which discussion and communication play such central roles. The role of the teacher as instructor, coach, mentor and learning participant is open for further investigation in the variety of different contextual settings in which the teaching of modelling and applications occurs.

The assessment issue seems destined to continue as a matter of importance and controversy. Rather than join the debate at the level of system requirements, a need exists for an epistemological approach to the question of validity and consistency. If modelling is a constructive activity, the constructivist paradigm will challenge any attempt to pretend that the ability of students can be validly assessed through performance on common problems set at national or state levels. Do we want our students to be good modellers, or is the purpose to achieve on some items chosen for test purposes? This dilemma is as alive now as it was two (or ten) years ago. Acceptance of inappropriate system assessment methods may be a currently imposed constraint; it must be not diminish efforts to win acceptance for alternative methods more faithful to the modelling enterprise.

As a coherent activity, the effective learning, teaching, and assessing of mathematical modelling challenges existing structures of mathematics and education. At the mathematical level debate continues within the academic and professional community about the inclusion of modelling – particularly in relation to formal content that might otherwise be covered. This of course focuses on uncomfortable questions, such as the purpose of learning mathematics! A banking metaphor is useful here. The conventional academic position is to provide bankable mathematics for up to 15 or 16 years of formal education. This knowledge is deposited but may only be accessed in particular forms, for example as cued by examination questions. In our teacher training programme we have found graduates unable to withdraw their "mathematical funds" for application to reasonably simple real life situations (except at a very basic level). More content would in no way facilitate this capacity which is related to a lack of 'infra-structure". Arguments for modelling challenge the citadel of institutionalized mathematics.

Beyond content itself, the organisation of learning, the processes used in teaching and assessing, and the independence granted and expected of students working on models, challenges many conventions associated with classrooms and schools. The choice of contexts to provide social critique rather than to confirm, for example, the values of consumerism challenge accepted structures beyond the learning context.

It is difficult to envisage any aspect of mathematics education that has the capacity to so coherently and comprehensively challenge existing structures as does mathematical modelling. It is therefore to be expected that opposition will be vehement and arise from many quarters if and when programmes in mathematical modelling begin to assert themselves beyond "acceptable limits". Perhaps we have not yet quite reached this critical point, beyond which programmes in modelling can no longer be tolerated as aberrations but are seen to demonstrably challenge vested interests in maintaining the *status quo*. One predictable reaction is the attempt to modify programmes (for example in scope or assessment) so as to maintain the form but to deny the substance. As a community we may expect to increasingly face subversive acts of this type.

Finally, the question of a substantive theory to support the teaching of modelling and applications remains essentially unaddressed. Modelling has a social content and occurs in naturalistic settings. Consequently theories need to be based on naturalistic paradigms. To this end grounded theory methodologies deserve attention as alternatives to the hypothesis testing approaches derived from positivistic science.

As to the ultimate challenge, the final word might be left to King Solomon

"Where there is no vision, the people perish."

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3

Modelling – A UK Perspective

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SUMMARY

This paper examines some of the major educational developments presently confronting the education community in the United Kingdom, and discusses them from the point of view of the teaching of mathematical modelling. It concludes that the time is right to open up mathematical modelling to other curriculum areas, and to seek to work much more closely with colleagues in all subject areas.

1. NATIONAL INITIATIVES IN HIGHER EDUCATION

The UK higher education system traditionally has been designed for the elite, with only a small proportion of those leaving school going on to university-level study. Recently this has changed dramatically and, between 1988 and 1993, the number of UK and EC students benefiting from higher education increased by 44% – to a population approaching 1 million. In the same period the participation rate of 18-19 year olds nearly doubled – from 15% to 28% – bringing the UK much closer to EC and international norms.

This system of 'mass higher education' brings its own challenges, and so the UK government has been funding a variety of development projects designed to raise awareness and expertise in relevant areas. These developments can be grouped into two main areas – those that involve assessment (such as projects on Learning Outcomes, Competences, Accreditation of Prior Learning, and Work Based Learning) and those that involve technology-based training (the use of new technologies in the teaching and learning process).

ASSESSMENT

Haines and Izard (1994) remind us that work on assessment has to be approached carefully - "...assessing the wrong thing better is of little use..." — and Niss (1993) gave us the interesting (almost associative) sum A

$$WYAIWYS + WYSIWYG = WYAIWYG$$

which shows that combining the everyday observation 'What You Assess Is What You See' with the well-known computer phrase 'What You See Is What You Get' leads to the illuminating 'What You Assess Is What You Get'.

Learning Outcomes are the traditional subject-based outcomes that are relevant in each individual subject area – what Izard (1994) might call "descriptors of student behaviour" – together with a range of personal outcomes that are often described as transferable skills and are applicable in a wide range of subject areas (see Table 1).

Subject-based outcomes	**Personal outcomes**
knowledge and comprehension	interpersonal skills
	(teamwork/negotiation)
applying knowledge in different situations	intrapersonal skills
processing skills	

Table 1: Learning Outcomes

The Learning Outcomes we are interested in when teaching Applications are probably on the left-hand side, while those for learning Modelling are more obvious on the right-hand side – an early indication that there may be very significant differences between the optimal approaches for what is often considered the single subject area Applications and Modelling (see Blum, 1991).

The literature of Learning Outcomes is still small, but a recent guide for staff (see Thorne, 1993) contains the chapter headings shown in Table 2. Again, these would be familiar to those of us designing courses on Mathematical Modelling, but would be far less relevant to courses on Mathematical Applications.

The behaviour we want from the learner
The degree of autonomy we want from the learner
The context in which the learning is demonstrated

Table 2: Writing Learning Outcomes

Learning Outcomes – and the related area of Competences and Work Based Learning, where we are interested in students demonstrating these outcomes in the workplace rather than merely in the classroom – are a necessary preliminary stage to work on assessment, since there is little point in improving the way that we assess the work of students if we are not clear exactly what it is we are wanting students to learn.

These issues become even more crucial when we begin the Assessment of Prior Learning (APL) – learning that has taken place prior to the student joining the institution. Sometimes that learning is Certificated (APCL), typically where the student has undertaken an incompany training course of some type, but more often (and leading to more difficulty in assessment) it is Experiential (APEL), typically where the student has learned a range of skills whilst in employment. I know of no work to date on APL in the area of Mathematical Modelling - perhaps because we are still unclear of the Learning Outcomes that we are trying to facilitate in the student!

TECHNOLOGY-BASED TRAINING

Teaching and learning is presently a big growth area in UK higher education, particularly the application of new technology to the teaching and learning process, but it is unfortunate that the use of technology-based training (TBT) is often driven by economic rather than educational reasons.

One of the more interesting reports in the area comes from the Scottish Vice – Chancellors (MacFarlane, 1992), whose main recommendations are shown in Table 3.

Title:	**Teaching and Learning in an Expanding HE System**
Recommendations:	Create Teaching and Learning Board
	Require universities to publish T&L strategy
	Funding Councils should take a strategic view of T&L in HE
	(motivation/initiative/self-reflection)

Table 3: MacFarlane Report

However, its recommendations may well lead to a higher level of standardization across UK universities which, while bringing resource savings, may tend to stifle the individual experimentation which is often the forerunner of exciting and worthwhile developments.

The national Teaching and Learning Technology Programme (TLTP) is also designed to lead to greater standardization, this time in the courseware that is used with students, but the approach taken has emphasized the importance of consortia of universities in increasing the local 'ownership' of the materials produced. The aim of the Programme is 'to make teaching and learning more productive and efficient by harnessing modern technology', and Table 4 gives details of timescales and funding available.

TLTP1	**TLTP2**
1992-93, for 3 years	1993-94, for 3 years
£7.5M in first year	£3M in first year

Table 4: Teaching and Learning Technology Programme

Table 5 shows a summary of projects funded under the first phase of TLTP, where work is being done on improving the learning technologies themselves, in specific subject areas, and – yet again – on the generic transferable-skill areas.

New technology	Generic areas	Subject areas
Multimedia	Study skills	Maths
Hypertext	Assessment	Stats
CAL		

Table 5: TLTP1 Projects

However, if we examine those TLTP projects in the area of mathematics (see Table 6) it is clear that they are all concerned with computer-based training in mathematical skill areas, and that no work is being funded in the area of Mathematical Modelling.

UK Maths Courseware Consortium
Coordinated development and evaluation of courseware for basic maths skills
Modules for remedial teaching of maths to scientists and engineers using Mathematica Notebooks
CALGroup Engineering Consortium

Table 6: TLTP1 Maths Projects

2. NATIONAL EDUCATION AND TRAINING TARGETS

Before moving on to describe some of the recent initiatives in schools, we ought to mention the ambitious National Education and Training Targets that were adopted by the UK government in 1991.

Table 7 shows the Foundation Learning Targets – those aimed specifically at young people. NVQ is the competence-based National Vocational Qualification awards scheme, where NVQ 2 is the level expected at the end of compulsory schooling at the age of 16 – for most students in England (Scotland having a separate system) this takes the form of the General Certificate of Secondary Education (the GCSE). NVQ 3 is the normal university entry level which, in England, would often be represented by the Advanced level exam (A-level).

Foundation Learning Targets	Lifetime Learning Targets
For young people	For the workforce
80% to NVQ 2 by 1997	All employees in training by 1996
NVQ 3 available to all	50% working for NVQs by 1996
50% to NVQ 3 by 2000	50% to NVQ 3 by 2000
Self-reliance, flexibility and breadth	50% of companies in Investors In People scheme

Table 7: National Education and Training Targets

Self-reliance, flexibility and breadth are all familiar Modelling aims, and attributes that we would encourage in all our students, but many UK observers have difficulty with the government's desire to demonstrate all these outcomes in the workplace – especially if they are observing from a school! Incidentally, the figures quoted in the Targets are approximately double the actual values that applied in 1991, so achievement of the Targets would represent a very considerable increase in the profile of education and training for young people.

Table 7 also shows, for comparison, the Lifetime Learning Targets – those aimed specifically at people in work. Once again NVQs are to the fore, as is the Investors In People programme – a scheme, including tax benefits, which is designed to encourage employers to value, and pay for, training. The Targets, as before, are approximately double the 1991 actual values, and so represent a very considerable increase in training in the workplace.

3. NATIONAL INITIATIVES IN SCHOOLS

Before discussing national initiatives in schools, it is worth reminding ourselves of the National Curriculum that operates in the UK – or rather in England, since Scotland, Wales and Northern Ireland operate somewhat different schemes.

Key Stage 1	Key Stage 2	Key Stage 3	Key Stage 4
Ages 5-7	Ages 7-11	Ages 11-14	Ages 14-16
Levels 1-3	Levels 2-5	Levels 3-7	Levels 4-10

Table 8: English National Curriculum

The curriculum is divided up into four Key Stages, at the end of each of which there is compulsory and standardised testing – at the ages of 7, 11, 14 and 16. These Standard Assessment Tasks (SATs) – rather similar to the Common Assessment Tasks carried out in Victoria in Australia – are presently the subject of much heated discussion in the UK, since classroom teachers are not convinced of the worth of the present tests and feel that a huge amount of class time is being taken up by an unreliable and untested instrument.

There is very little mention of Modelling in the Mathematics sections of the National Curriculum (NC), and one has to look in the IT section of the Technology NC in order to find the headings given in Table 9.

IT in the National Curriculum
Communicating information
Handling information
Modelling
Measurement and control
Applications and effects

Table 9: IT in the National Curriculum

Aims such as 'communicate and handle information', 'design, develop, explore and evaluate models of real or imaginary situations' and 'make informed judgments about the applications and importance of IT and its effect on the quality if life' are familiar to those of us working in Mathematical Modelling, and a welcome addition to the NC – even if somewhat buried in all the verbiage!

A recent report by Her Majesty's Inspectorate (HMI, 1988), when examining IT in secondary schools, noted that

> A characteristic of much work in IT is that pupils work for prolonged periods without needing support, encouragement or prompting

which is echoed by many observers of modelling work in schools (see, for instance, Galbraith, 1994). However, when the National Curriculum Council carried out a series of monitoring studies, they found that only about 10% of the pupils were actually carrying out work on Modelling and only about 10% were working on Applications (see Table 10, which is taken from the same HMI Report).

Key Stages 1 & 2	For IT Attainment Target
65% Year 1 used IT for at least 1 hr/wk	50% handling information
50% Year 3 used IT for at least 1 hr/wk	50% communicating information
33% of classes had exclusive use of a PC	10% modelling
<40% of Year 1 used a printer	20% measurement and control
<40% of Year 3 used a concept keyboard	10% applications and effects

Table 10: NCC monitoring of IT in schools

One possible explanation could be that there is still little good material on these areas in the Technology literature, and that teachers are unwilling to move into new areas without adequate support. However, there is now a substantial body of material in the area of Mathematical Modelling, much of which would be transferable immediately to this new arena if publicised appropriately. Here we have a clear example where subject boundaries are inhibiting progress on the wider front.

Another interesting point made in the HMI Report concerned equal opportunities.

Statistics	More boys than girls use computers Parents more likely to buy home computer for boys Computer games aimed at male market
Technology and jargon	Boys see computing as an interesting hobby
Teachers	Many secondary IT teachers are male
Classrooms	Boys dominate computer activities
Groupwork	Girls work better in cooperation, rather than competition

Table 11: IT and Equal Opportunities

I am increasingly concerned by the preference shown by boys and young men towards Modelling, and the fact that this added motivation sometimes leads to better performance than by girls and young women. This might reflect the increasing use of IT in Modelling – and hence be a direct mirror of the effect observed by the Inspectors – or it could be one of the more subtle problems discussed by Blum (1994). Clearly more research is needed in this important area.

The Department for Education has recently funded several projects in the area of information services one of which, promoting the use of CD-ROMs in schools, is outlined in Table 12.

Run by:	National Council for Educational Technology
Funding:	£2M from Department for Education
Scope:	About 600 secondary schools - 1 CD-ROM drive and 4 CD-ROMs
Discs:	Most schools used same discs ECCTIS Grolier Times/Guardian/... Shakespeare

Table 12: CD-ROMs in schools

The CD-ROM in Schools project provided funds for one CD-ROM drive and four discs (mainly information services such as newspapers and encyclopedias) in all secondary schools and a few primary schools. The evaluation study makes interesting reading (NCET, 1992), and shows that a dramatic effect has been made on many subjects – areas such as English and History, for instance – but that there has been little discernible effect on Mathematics or on Modelling. This seems strange, since so many Modelling case studies are based around data handling and information, but the materials are so new that time is perhaps needed in order for them to be widely used right across the curriculum.

4. INITIATIVES IN THE HOME

When compared with the £2M spent on the (very successful) CD-ROM in Schools project, the amount spent on computer games each year is clearly of a different order of magnitude – almost £700M in the UK alone last year!

Heppell (1993) made an impassioned appeal to all schoolteachers to review their attitude to home computer games – the sort manufactured by Nintendo and Sega – since we would dearly love children to be 'hooked' on learning at school in the same way that they are on computer games at home, and the negative attitudes displayed by many adults – ourselves, as educationalists, included - only increases the gulf between learning and playing.

When Sega ran a competition, last year, to design a television advert for them, no less than 700 schools took part. It would have made an ideal Modelling project, integrated into several streams of the National Curriculum, but unfortunately very few schools took this opportunity to bridge the gulf between home and school.

On one level this is entirely understandable. More children in the US now recognise Mario – and his mates Koopa, Troopa, Bloober and Podoboo – than recognise Mickey Mouse, and I sympathise with children who find Mario and Sonic more appealing than learning sums, so what should we do?

If we try to categorize computer games into various styles, we find that they tend to fall into the following categories.

Games styles	Features	Attributes
Narrative	Watch and learn	Observe
Interactive	Choose and so	Question
Participative	Contribute and create	Hypothesize and test

Table 13: Games styles, features and attributes

Really successful games tend to be Interactive and Participative, just like good teaching! They also seem to involve the sort of attributes shown which are, yet again, just the sort of thing we encourage in Modelling classes.

Progress cannot be made until we appreciate our prejudices: not all computer games are bad – nor are they all good – but the problem-solving skills that many successful games display are transferable to other areas of children's learning. As Heppell puts it, "The Andy Pandy generation is leading the Sonic generation into the information age – and the Andy Pandy generation has some homework to do".

Many of these points are really to do with a change from teaching to learning – what Heppell describes as "The Sage on the Stage making way for the Guide on the Side" or Galbraith's (1994) description of the teacher as "The More Capable Other – but there is also an underlying message. Hitherto we have seen Information as the preserve of Mathematics, but other subject areas are now asserting their rights to access Information directly - and Mathematics is in danger of becoming marginalised if this attitude is not re-examined.

5. THE LINK WITH MODELLING

Many of the comments above may appear to have little to do with Mathematical Modelling – 'Modelling' has been mentioned frequently, but 'Mathematical Modelling' appears only occasionally. I would like to pose a few questions for the Mathematical Modelling community.

- *What are we teaching – and why?* The use of learning outcomes and capabilities may help us answer this vital question – and to distinguish between Mathematical outcomes that we are attempting to facilitate, and more general transferable skills.

- *How will students use what we teach them?* The study of competences and vocational relevance may help us to focus our attention on what is needed and what is peripheral.

- *What is the place of the new learning technologies that are now available?* New technologies can help us minimize the maths – and so allow students to focus attention on the modelling – but we must be careful when adopting this approach.

- *Is our primary interest in Mathematics, or are we really concerned with a range of Information skills that are of much wider interest?*

As a postscript, I would like to refer to the only series of conferences that is dedicated to the teaching of mathematical modelling – the ICTMAs. ICTMA-1 and ICTMA-2 were entitled International Conference on the Teaching of Mathematical Modelling (perhaps they should have been ICTMMs?), while ICTMA-3, ICTMA-4, ICTMA-5 and ICTMA-6 were all entitled International Conference on the Teaching of Mathematical Modelling and Applications (surely ICTMMAs?).

Perhaps the communal subconscious of the organisers is at work here, and there really will be an ICTMA soon – an International Conference on the Teaching of Modelling and Applications – concentrating more on transferable skills, the social uses of Modelling, the history and the philosophy, and less on its birthplace within Mathematics? Only time will tell.

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Section B

Empirical Investigations

4

Developing Metacognitive Skills in Mathematical Modelling-A Socio-Constructivist Interpretation

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SUMMARY

Research has shown that knowledge alone is not sufficient for successful modelling: the student must also choose to use that knowledge, and to monitor the progress being made. Metacognition involves awareness and control of one's own thinking. Distinct elements in metacognition have been identified, including knowledge of one's own thought processes, the control and application of that knowledge, and the learners' beliefs about the nature of mathematics and themselves as learners.

This paper describes an action research project into the use and practical application of mathematics. The learning of mathematics is interpreted as a socio-constructive process. Modelling skills were found to be most successfully acquired when 'scaffolding' is provided to structure the task into steps that the student can just manage. Questions and discussion which form this scaffolding should be general rather then task-specific to facilitate their internalization into student's schema.

Effective teaching approaches are identified which socialize students into a consensual interpretation of modelling culture. This supports

the development of the metacognitive skills of planning, monitoring and evaluating which are integral to successful modelling.

Developing Metacognitive Skills In Mathematical Modelling: A Socio-Constructivist Interpretation

INTRODUCTION

The Use and Practical Applications of Mathematics Project was funded by the Welsh Office during 1991/92. The project aimed to develop approaches and materials to teach and assess the thinking skills involved in using and applying mathematics in practical, modelling situations, with students aged between 11 and 16.

Problem solving and modelling are of increasing importance in mathematics. The national curriculum for mathematics in England and Wales requires all students to use and apply their mathematics in a variety of situations including practical tasks and real life problems. Since the 1970's, mathematical modelling has been seen as a unifying theme for all applications of mathematics (Burghes, 1980) and calls are growing for modelling to form an explicit part of the mathematics curriculum (Blum and Niss, 1991).

The terms *mathematical model* and *modelling* are no longer restricted to concrete geometrical objects and their construction. Recently, much wider definitions have been employed:

> The process of starting with a real problem, abstracting and solving a corresponding mathematical problem, and then checking its solutions in the practical situation is often called mathematical modelling. (DES, 1985 p.41)

The stages and processes involved in mathematical modelling have been amplified into flowcharts or algorithms by several researchers (eg: Swetz and Hartzler, 1991). Mason, (1988 p.209) provides a basic framework for analysing the modelling process:

1. Specify the real problem
2. Set up a model
3. Formulate the mathematical problem
4. Solve the mathematical problem
5. Interpret the solution
6. Compare with reality
7. Write a report

While these models of the process may be useful in describing modelling to beginners, they are less satisfactory as models of the processes which learners go through when learning to model. All these models may be criticized for the false dichotomy they create between an individual's perception of reality and the mathematics which s/he has previously constructed. The form of model which a learner develops depends on both their pre-existing mathematical constructs and the social context in which the learner is situated. The reality with which a model is compared may be a social reality rather than a physical one. Mathematical knowledge is a necessary but not sufficient condition for successful modelling.

THE ROLE OF METACOGNITION

Research has shown that knowledge alone is not sufficient for successful modelling: the student must also choose to use that knowledge, and to monitor the progress being made (Silver, 1987). It has been argued that explanations of problem-solving difficulties based on purely cognitive factors are incomplete (Lester, 1987), and attempts have been made to integrate metacognition into theories of modelling and problem solving (Schoenfeld, 1987).

Metacognition involves awareness and control of one's own thinking (Brown, 1987). Distinct elements in metacognition have been identified, including knowledge of one's own thought processes, the control and application of that knowledge, and the learners' beliefs about the nature of mathematics and themselves as learners (Lester & Kroll, 1990).

Gray (1991) identified three strands in metacognition which support the modelling process: planning, monitoring and evaluating. During modelling, mathematicians alternate between these strands, continually suggesting ideas and strategies, evaluating and criticizing them, and monitoring the progress made. That is, expert modellers are able to discuss and argue within themselves (Schoenfeld, 1987; Wheatley, 1991). Students need to develop such metacognitive skills of 'inner speech' in order to become effective modellers.

PROJECT METHODOLOGY

A network of eight secondary schools was established. The action research paradigm was chosen due to the novelty of the teaching approaches which were being developed. Lessons were monitored through self-evaluation forms and by university researchers who acted as participant observers in approximately 100 lessons. No attempt was made by the university researchers to be 'invisible' in the classroom, rather we regarded ourselves as full participants in the experience and interfered significantly with the processes we were observing. By asking students to describe the strategies they were using and how these had been selected, we placed emphasis on planning, monitoring and evaluating. Most teachers asked similar questions themselves.

The teacher researchers attended regular network meetings to discuss progress and to develop approaches and activities. The activities were designed to allow students to pose their own problems from real world situations and formulate and test their own mathematical models.

The focus of attention of the university researchers was the interaction between participants in the lessons and the way in which these interactions affected the development of teaching strategies, consensual perceptions of modelling and modelling strategies, and individual problem solving and modelling skills.

HOW DO CHILDREN LEARN TO MODEL?

Much of the research on problem solving and modelling can be criticized for ignoring the role of the teacher (Silver, 1987). While the current dominant paradigm for the learning of mathematics is one of constructivism (Cobb *et al.*, 1992), the social nature of learning should not be overlooked:

Mathematical learning can be viewed as both a process of individual construction and as a process of acculturation into the mathematical meanings and practices of wider society. (Eisenhart, 1988).

Ernest (1991) claims that mathematics is a social construction based on linguistic knowledge in which objectivity is gained through publication, public scrutiny and criticism. Objective knowledge, therefore, is a social construct which is continually created and recreated. The teacher has to create such a climate of mathematical discussion and challenge within the classroom to enable students to construct and test their ideas explicitly with others.

Vygotsky (1978) suggests that a child learns by interacting with more capable others who provide sufficient support for the task to be completed. The teacher acts as 'a vicarious form of consciousness' (Bruner, 1985 p.24), structuring tasks and controlling the path of solutions until such time as the child achieves conscious control of a new function or conceptual system. Vygotsky viewed such internalization as a social process mediated by language, with external speech used for communication with others and inner speech for planning and self regulation.

Hirabayashi and Shigematsu (1987) argued that students develop their concepts of metacognition by copying their teacher's behaviour, and thus their executive or control functions represent an 'inner' teacher. Vygotsky (1978) suggested that all such higher order functions originate as actual relationships between individuals, thus before students can 'internalize' these skills they must develop them explicitly with others. Discussion and questioning within a supportive group leads students to construct a 'scaffolding' framework for each other, which enables them to solve problems collaboratively before they can solve such problems individually (Forman and Cazdan, 1985).

The teacher has a pivotal role in helping students to learn mathematics. Mathematical discussion between students has to be facilitated while constraining their interpretations and solutions in harmony with those of the wider mathematical community. This must be undertaken in a communicative context that involves the explicit negotiation of mathematical meanings (Cobb *et al.*, 1991). Any exploration of the development of students' modelling abilities must, therefore, consider the social interactions that occur within the learning environment.

APPROACHES TO THE TEACHING OF MODELLING

A variety of teaching approaches were developed and piloted by the action research network of teachers and university researchers. We have characterised these under six broad headings although few teachers restricted themselves to one approach.

Sink or Swim: Students were *thrown in at the deep end* with little or no guidance beyond a simple unstructured worksheet. Inexperienced students were sometimes left floundering, unsure of what was expected of them and lost as to how to proceed. This technique was successful only with the most able students and even here it was not the most efficient. Pedagogy based on constructivism is often problem-solving centred and, in some characterisations, has led to the belief that mathematical learning should be "a process of spontaneous, unguided, independent invention" and the "indefensible" belief that teachers should not intervene. (Cobb Yackel & Wood 1992 p.27)

Cookbook modelling: Students were led to a strategy either through a highly structured worksheet or through strong teacher direction. This was superficially attractive in that students were maintained *on task* easily, but in the longer term the approach proved disappointing in that metacognitive skills were not developed. Although aspects of the structure could have been generalized to other situations, the teacher did not emphasise them. Students were left to identify and abstract these elements for themselves. Modelling skills which had been "taught" using this technique failed to transfer to even slightly different situations.

Questioning Using Organisational Prompts: A list of organising questions was given to the students and was referred to over a range of activities. The list formed a strategic planning structure which was used as scaffolding by students during the early stages. It was supplemented with oral questions which were asked on a regular basis, eg: 'Can you explain your plan to me?', 'Does that always happen?'. The aim was to encourage students to develop a framework of questions to organise their thoughts. An expectation developed that such questions would be asked and students seemed able to internalize them for use in planning.

Internalization Of Scientific Argument: Groups of students were required to present interim approaches and findings. Presentations were followed up by questioning and debate. Questioning was led by the teacher at first, with a gradual increase in the amount initiated by students. Students began to copy the form of question used by the teacher when framing their own. It became clear that groups were anticipating the same form of question about their own presentation and preparing a suitable response. The students were learning how to conduct a scientific discussion (Wheatley, 1991) and 'argue with themselves' (Schoenfeld, 1987). Small group processes such as explaining ideas, challenging the ideas of others, and reaching a consensus, lead to structural reorganisation, more meaningful cognitive elaborations, and the induction of reflective thought (Noddings, 1985).

Start, Stop, Go: Tasks began with a few minutes of silent reading and planning. Small groups then discussed possible approaches. A whole class brainstorm followed before returning to small group planning. This ensured that all students engaged with the task and began to plan, but that a variety of perceptions and plans was examined and evaluated. At intervals the class was stopped for reporting back. Students began to anticipate not only the form of questioning which would be used, but also that reporting back would occur. Groups began to monitor their progress in anticipation, which restrained impulsive planning and encouraged self-monitoring.

Research has shown (Schoenfeld, 1987) that good modellers are able to work in a strategic or control mode and thus monitor their progress at each stage, whereas novices proceed through a problem one step at a time without checking for sense. Successful modellers take an overview of the situation before deciding on one approach from several whereas novices tend to divine one approach and follow it without reflection. The "Start, stop, go" approach emphasizes the need for self-monitoring and reflection.

Using Peer And Self-assessment To Encourage Reflection: Students were required to write up their work individually, but selected groups also presented their final report to the class for peer assessment. A vital question which had to be answered was: 'If I were to do this investigation again what would I do differently?'. This encouraged 'looking back' and allowed consideration of the elegance of different solutions, economy of approach to a task, etc. Assessing the work of others helped students to negotiate and objectify the nature of a good solution, and reporting back made them identify what was important in their own work. Reflecting on the work of others led students inevitably to reflect back on their own work. Through assessing the work of others, students learned to evaluate and regulate their own thinking.

Students were encouraged to assess their own work against a self-assessment framework for each activity. This formed the basis for a dialogue between the student(s) and the teacher which helped them to understand the criteria against which they were being assessed.

ACCULTURATION INTO MODELLING

Supposedly 'open' situations are not as open and free as they seem. To learn mathematics is to take part in a process of socialization, or acculturation (Schoenfeld, 1987). When students approach a task, many of the decisions about processes and outcomes have already been determined by the common assumptions of the classroom environment.

It is sometimes argued that it is not possible to assess open ended work due to the variety of outcomes which might be found. This implies that a good solution would be impossible to recognise when presented. This is not the case. Good solutions are 'reliably recognizable' (Scriven, 1980). Mathematicians work to a set of assumptions, often unstated, relating to generality, economy and elegance. Teaching students mathematics must include socializing them into these assumptions.

The explicit negotiation of meaning that occurs during 'scientific argument' and 'reporting back' enables students to appropriate the values of modelling culture. The teacher mediates and guides this process, emphasising those aspects which are judged to be significant for the students' future learning. The small-group discussions during 'start, stop, go' empower the students to participate in this process of constructing modelling culture. Students are encouraged to test their individual, subjective constructions through discussion and comparison with those of others. Subjective constructs gain validity through social acceptance, thus developing students' concepts of justification and proof.

To be able to operate in an open, practical or modelling situation, students must be aware of the general nature of qualities to be found in a desirable solution and be able to assess their own work against these criteria. The process of peer assessment negotiates a consensual interpretation of modelling culture and self assessment involves students in testing their individual constructs against the constructions of others. Involvement in peer and self-assessment helps to develop the metacognitive skills of monitoring and evaluation.

CONCLUSION

Learning is a process of individual construction constrained by social interactions. Teaching modelling must socialize students into a consensual interpretation of modelling culture. The metacognitive skills of planning, monitoring, and evaluating are integral to successful modelling. These skills are best developed through the provision of scaffolding, through questioning to promote the internalization of organisational prompts, and scientific argument using the technique described as 'start, stop, go'.

Participation in peer and self-assessment involves the student in a recursive, self-referential learning process which supports the construction of increasingly sophisticated mathematical concepts. Self-assessment is at the heart of the modelling process and learning the skills of self-assessment helps students to learn to model.

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5

Cognitive Processes and Representations Involved in Applied Problem Solving

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SUMMARY

The main purpose of this paper is the presentation and discussion of the data obtained through the observation and analysis of an activity of applied problem solving. We have in mind the identification of the conceptual models that students activated and made operational in the process of modelling a real situation. To achieve this, we considered the combination of two levels of analysis: the dynamic features of the process, concerning the interplay between mathematics and the real situation, and other more static elements, namely the delimitation of phases in student's work.

1. INTRODUCTION

One of the nine applied problems used in the course of the research project Modelling in Mathematics Teaching 1 was the paper roll problem. Many different references to the discussion and presentation of this problem can be found in the literature. We adapted a version of the problem given by Swetz and Hartzler (1991). A set of questions concerning this problem were presented in a worksheet to a class of 10th grade students (see Appendix 1). Students worked on those questions in their regular mathematics class for about 90 minutes. Computers with electronic spreadsheets and some toilet–paper rolls were available. One group was observed and their activity was fully recorded with a movable video camera.

The isolation and characterisation of the most prominent phases of student's activity, during the exploration of the paper roll situation, assumes a form of *outlining* that can induce and facilitate a deeper analysis of the cognitive processes developed.

The *outline* presented here reflects essentially two aspects:

(a) the evolution of student's conceptual models (interpretations) of the situation;

(b) the connections "mathematics $\leftrightarrow$ real situation", displayed in student's activity, in terms of transference processes from the real situation to mathematics and conversely.

2. PHASE I–PERCEPTIONS OF INTUITIVE NATURE; THE IDENTIFICATION OF VARIABLES.

After having recognised the need for additional information–the thickness of the paper–students delineated a process of "counting" to describe how the radius of the roll would increase as the paper was being rolled up. One of them, taking 0,1 cm for the thickness of the paper, explained:

–"If it [the radius] is 5 in the first wrap, in the next one it will be 5.01, and in the next one it will be 5.02..".

It should be pointed out that this procedure, simultaneously *additive* and *recursive*, is very easily transferable to the spreadsheet, by means of entering a formula that relates one cell to the previous one in a certain column. This perceptual system became fairly consensual in the group. Almost immediately, students seemed to feel the need of calculating the length of paper rolled up in each wrap. One of the students wondered about what should be appropriate to use at that stage: either the formula for the area of a circle or the formula for the perimeter. Eventually students decided to use the formula for the perimeter of the circle in their following investigations.

A member of the group suggested the use of the spreadsheet, saying that they should define a column for the radius of the paper roll in each wrap and another column for the length of paper rolled up in each wrap. We find plausible that the formula for the perimeter of a circle has prompted the radius of the roll a significant independent variable.

In this first phase, we believe it is possible to identify two kinds of conceptual models of the situation: a model qualified as *additive* and *recursive*, followed by a *functional* model. In both cases there are indicatory clues that students' perceptions arose from different reference systems of the situation.

In the first model, we may devise a process developed upon intuitions. The action of rolling up the paper again and again was associated with the repeated addition of a certain constant value, identified as the thickness of the paper. This leads us to the idea of a translation from reality to mathematics, here represented as *real situation→mathematics.*

In the other conceptual model, students interpreted the real situation according to a certain geometrical model. It is clear their conception of the paper roll was a sequence of concentric circumferences. There was the capture of real elements and their integration in a mathematical framework, showing a translation process of the kind *real situation→mathematics.*

Later on, the formula for the perimeter of a circle showed the length of each wrap as being dependent on the radius of the roll, which determined the awareness of a significant variable. As students transferred this particular mathematical relation into the context of the paper roll, they gained a new perception of the real situation, namely the idea of a connection between the successive radii and the successive wraps' lengths. It seems adequate to look at this particular aspect of students insight as a translation *mathematics→real situation.*

3. PHASE II - STRUCTURING MATHEMATICAL RELATIONS; THE SEARCH FOR ANSWERS.

Students enthusiastically started to create a numerical table on the spreadsheet as soon as they became confident on the possibility of computing some of the variables involved. The first version of their table showed three columns in a sequential order representing, respectively: <u>number of wraps, radius, perimeter</u>. This particular kind of structure, due to the relational nature of the spreadsheet, encapsulated the composition of functions:

number of wraps → radius → perimeter.

As a result, students obtained a representation of the relation between the wrap's number and the length of paper rolled up in that wrap.

This phase also included the process of getting a formula to determine the total length of paper rolled up as the number of wraps increased. Once again, the model activated had an *additive* and *recursive* nature: one has to add the perimeter of a certain layer to the total length rolled up until that layer. The creation of a new column on their table captured this very intuitive nature of their computation model and made it operational through the use of the spreadsheet.

The following step was the search for an effective answer regarding the length of paper existing in a roll with an internal radius of 5 cm and an external radius of 18 cm. Students used their spreadsheet data as a way of applying their model to the real problem. They considered R=18 as the radius of the last wrap and looked on their table for the corresponding total length of paper which was 47.69 meters (see Appendix 2).

In reviewing student's activity, we find that they turned themselves once more to the real imagery of rolling up the paper to determine the total length of paper rolled. They got a form of computing it by means of a recursive model. It shows a new shift *real situation→mathematics* in their thinking process..

Also, in the table produced, each wrap was associated with two different variables: the length of paper rolled up in that wrap and the total length of paper existing in the roll. This representation of the real situation turned out to be rather solid in students' posterior thoughts as if it became an embedment of reality. It presents another instance of the transference *mathematics→real situation* in student's cognitive processes. This kind of transference became even more distinct when students looked for the the total length of paper rolled up, using specific results presented in their table (what happens when R=18?). Students applied their model to a particular situation in order to discover the necessary answer by means of their computational representation of the situation.

4. PHASE III - THE SPREADSHEET AS AN INSTRUMENT OF VALIDATION OF AN ANALYTICAL MODEL

In the worksheet (see Appendix 1) it was suggested that students prove that the number of wraps in the roll could be given by the equation: $N = (R - r)/t$.

For a while, they were detained in the interpretation of the meaning of the variables in the equation and they looked for connections with the elements shown in their table. They finally decided to use the spreadsheet as a calculator and computed the number N for the given values of r and R. The result turned out to be 65 wraps. But when they looked at their table they noticed that the external radius, R=18, was paired with the value 66 in the column for the number of wraps (see Appendix 2). This contradiction was responsible for students' suspicion about the given equation. However, their next step was to find the possibility of "fitting" the formula to their own data. They thought they should translate it into a new column in the spreadsheet. Then, to get the first value of N equal to 1 (the first wrap), they came up with the recursive formula $N = (R_{N+1} - r)/t$. When they copied this formula along the new column up to the row that contained the radius of 18 cm, they found that the last value of N was negative. Curiously, they started to formulate conjectures based on the real situation to explain why that negative number had showed up. One of the students presented the following argument: "It's when the paper is over. It makes all those layers and there is lack of paper for the final wrap". This comment was accepted by the other members, indicating that students were inclined to agree on the existence of 65 layers in the roll. As they stated, the 66th wrap was not actually performed because the paper was over.

In this phase the main stream of students' activity concerned the use of the spreadsheet as an instrument to validate (confirm) a certain mathematical relation about the paper roll. They dètected an incoherence between the numerical results produced by the given equation and their own spreadsheet data (65 versus 66 wraps). However, they looked for a way of adjusting the given mathematical model to their own sense of the situation. The significant detail of ignoring the first value of R ($R_1 = 5$) in the recursive formula to compute the variable N shows their effort to adjust the outputs to their own data. We consequently consider to have happened a transference *real situation→ mathematics.*

When students finally faced the negative last value of the variable N, they had to deal with a new mismatch between their previous data (their image of reality) and the formula output. The relevant finding to this matter is that students chose to search reality to explain that particular result. The integration of aspects imported from the real situation (there is no paper for the "last" wrap) into their reasoning, as a way of making sense of the result $N < 0$, finally led to the rejection of a total of 66 wraps. Therefore we may understand it as a transference of the type *mathematics→real situation.*

In spite of these various connections between mathematics and the real situation, students did not criticize the validity of their basic representation of the roll. They never came back to their former result for the total length of paper rolled up (which included an extra wrap in the roll) or tried to question and to improve their own model.

5. PHASE IV - THE EXPLORATION OF THE MODEL; GRAPHICAL REPRESENTATIONS

During this phase, students worked on questions of the type "what happens if..". They began by studying the effect of duplicating the thickness of the roll on the total length of paper rolled up. Their first move was based on the real situation and they elaborated on mental images of the problem. They assumed that the total length of paper had duplicated and tried to work out if the number of layers would also duplicate. Their conclusion was that the number of layers would not increase to its double since the successive layers would be longer and longer. Afterwards, they came back to their spreadsheet and extended their table until the number of wraps was duplicated. They verified that the total length of paper in that case was not the double of the value found before.

This problem was also the motive for a subsequent debate about the form of variation of the total length with the radius of the roll. After a suggestion of the researcher, they made its graphical representation on the spreadsheet. The graph produced showed the plotting of the number of wraps against the total length of paper (see Appendix 2). Students made some interesting comments as they interpreted the graph. They kept in mind the idea of rolling up successively the same amount of paper and saw how the correspondent number of wraps was getting smaller. A detail of some relevance at this stage is that students introduced a rectangular grid in the graph. It then showed clearly the representation of equal increases in the total length (in the y–axis) and the correspondent number of layers (in the x–axis), which was decreasing.

Once more, students' work appeared to be heavily centered on their perceptions of the real situation. One of the most relevant issues concerns the way students analysed the relationship between the radius of the roll and the total length of paper. They made no attempts to describe it by means of an algebraic expression. On the contrary, it all seems to have been anchored in a certain vision of the real situation, by exploring the idea of continuing to roll up the same amount of paper in the roll. Student's first conclusion about the question of duplicating the radius appealed to the real situation and took the form of a mathematical result: the length does not duplicate when the radius does. We think that it fits our idea of a transference *real situation→mathematics*. Nevertheless, this interpretation was confirmed on the spreadsheet in a reciprocal way. The extension of their table was used to verify what should be the length of paper if the radius of the roll would duplicate. In what concerns our shifting categories, we can detect a new translation *mathematics→real situation*, with students applying their own model to confirm the answer. This bilateral view of the problem seemed to have been the result of integrating a certain mental image of the situation with a functional representation presented in the spreadsheet table. We also find it significant that students reactivated their mental image of the duplication of the length of paper as they reflected upon the graphical representation produced. It may indicate that the idea of fixing a certain amount of paper to be rolled up successively and then see what happens to the number of layers, persisted rather stable. Somehow, the use of a grid in the graph may have helped to prompt that intuitive perception of the real situation within a new representation. Therefore, it is possible to conclude that the double connection between the real situation and mathematics was maintained after the construction of a new graphical representation of the problem.

6. STUDENTS' COGNITIVE PROCESSES LINKED TO THE MODELLING ACTIVITY

One of our main concerns was the clarification of the nature of student's modelling activities as we intended to construct a certain mapping of what was evolving from a cognitive point of view.

It is very common to assert that a mathematical model consists of a "mathematical representation" of a certain phenomenon or real situation. Usually connected to that idea is the assumption that a mathematical model is usually expressed by an equation or a system of equations (sometimes being differential equations) that relate the variables identified in the problem and describe their behavior (Ogborn, 1991; Winkelmann, 1991).

As regards this theoretical perspective we find it useful to make notice of two interesting contributions given by Niss (1989) and Swetz (1989). The first of these authors conceives a mathematical model of a real situation as a triple (A, M, f), where A is a certain context or domain of reality (also designate by extra-mathematical), M is a collection of mathematical objects, relations and structures and f is a mapping which enables the translation of elements of A into elements of M. This put before us a conception of mathematical model where reality and mathematics are interacting parts of a whole. This interaction should be seen as a dynamic process. The mathematical model should not be elected as a 'portrait of reality', neither should it mean a statical condensation of a piece of reality in a certain system of formal language.

On the other hand, Swetz points out an aspect that seems to be of undeniable importance when we want to value the interplay between mathematics and reality in mathematics teaching. According to his view there are multiple forms of revealing the articulation between mathematics and real situations, many of which are already consigned in the traditional school curricula. He refers to things like the organization, construction and analysis of tables of data, to the creation and interpretation of graphs of functions, to the manipulation of equations and inequations, to the use of matricial representations, to the elaboration and application of algorithms, including those that can be used in computers, etc. This shows an appeal to several forms of mathematical representation in the process of modelling a real situation, a concern that is highly consensual between experts in this matter.

If we decide to combine both these views, we will get to the argument that a mathematical model sustains a permanent interaction between reality (A) and mathematics (M) as well as it calls upon a diversity of mathematical representations to describe that interaction. It is based on this notion of mathematical model that we find the cognitive processes of students to be essentially characterised by this dynamic balance between their reasoning about reality and their mathematical ideas and concepts. There is no doubt that the translations between mathematics and the real situation were abundant and developed in both ways, being the sign of an existing flow of modelling connections. The aspects of the real situation under analysis changed in the course of students activity. Also the mathematical elements activated in each phase were diverse. But the main issue is that students' processes throughout their work showed a common trace: *the dialog mathematics–reality.*

In trying to capture the essence of students' thinking processes we became aware of the permanent flow that was going on during the activity. We then explored the possibility of recognizing students mental processes as a succession of transfers between mathematics and the real world and we tried to illustrate how these were developed in both directions–from the real situation to mathematics and vice versa. In doing so, we came closer to the idea that the whole modelling activity has a cognitive architecture that could consist of a multiplication of *micro-modelling cycles.* In this kind of perspective much importance is gained by the *coordination* of the two *sources* (mathematics and reality) in feeding students' reasoning about the problem situation. We find plausible that students' understanding of the problem situation is developed as far as they build connections between some aspects of the real context and some elements of mathematics and as far as they make sense of them (that is create or recreate their conceptual models of the situation).

7. THE (MULTI)REPRESENTATIONAL NATURE OF STUDENTS' MATHEMATICAL MODELLING

In light of what has just been said, most of students' representation processes were conformed with two general reference systems. One refers to what students perceived from the real situation, according to their own experience and knowledge of specific aspects involved in the problem. For instance, during the whole activity the prevailing image of the roll was a succession of concentric circles representing the layers of paper rolled up. The other reference system included ideas, concepts and mathematical procedures, most of them tuned with the computational instrument they used. This double nature of their interpretations of the problem is consistent with an idea supported by

Lesh (1990) according to whom applied problems are "multimodal", as they appeal to and stimulate the use of several representation modes.

These two main guidelines of students' activity are still complemented by the multiplicity of mathematical representations involved in the evolution and exploration of their models. They used a variety of representation forms, between verbal statements relating variables and describing patterns of behavior, the production of formulas (computation algorithms), and the creation of tables in the spreadsheet and graphical representations. Again we may recall that the algebraic representation was never tried or even felt necessary by the pupils.

We can say that the computational representation, attached to the spreadsheet, has basically dominated students' performance. It could also be said that this computational representation has influenced the form of attacking certain questions (as in the case of the validation of a given model). Moreover it may have produced a *freezing* effect in the vision of the paper roll, since students never questioned it or even compared it with the real toilet paper rolls they had at hand.

Note 1. The Project MEM (Modelling in Mathematics Teaching) is an on–going project funded by IIE (Instituto de Inovao Educacional) and developed by a group of researchers and teachers since 1991. It's a three-year project concerning the investigation of student's processes on modelling and applications, the curricular integration of such activities, and the use of computers for modelling purposes.

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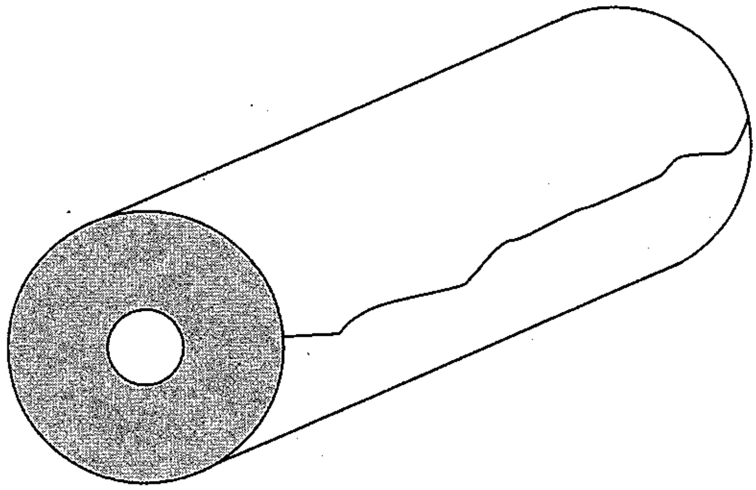
APPENDIX 1
PAPER ROLLS

Many people use paper rolls in the kitchen to dry their hands. The thickness of the paper varies with the type of roll, but in all of them there is an inner cylinder of card around which the paper is wrapped. The radius of the inner cylinder of the roll ROLA is 5 cm and the external radius (cylinder plus the paper) is 18 cm.

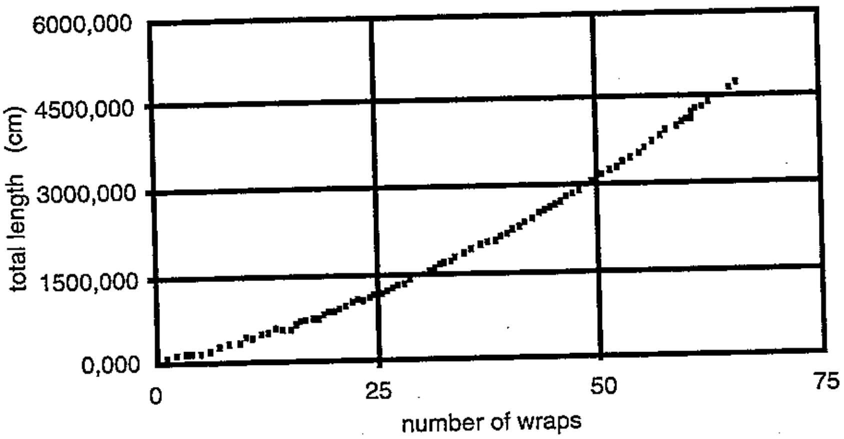


1. Which data is needed to know the length of paper stored?
2. Construct a mathematical model to calculate the length of paper existing in any paper roll, whatever may be its mark.
3. Show that $N = (R - r)/t$, where N represents the number of wraps, R is the outer radius, r is the inner radius and t is the thickness of the paper.

(Adapted from Swetz & Hartzler, 1991, *Mathematical Modelling in the Secondary School Curriculum*, p. 55–59. See references.)

APPENDIX 2

int. radius	5			
ext. radius	18			
thickness	0.2			
				Formula
wrap	radius	perimeter	total length	N = (R–r) / t
1	5	31.416	31.416	1
2	5.2	32.673	64.088	2
3	5.4	33.929	98.018	3
4	5.6	35.186	133.204	4
5	5.8	36.442	169.646	5
6	6	37.699	207.345	6
7	6.2	38.956	246.301	7
8	6.4	40.212	286.513	8
9	6.6	41.469	327.982	9
10	6.8	42.726	370.708	10
11	7	43.982	414.690	11
... ...	... ...	... ...	... ...	... ...
51	15	94.248	3204.425	51
52	15.2	95.504	3299.929	52
53	15.4	96.761	3396.690	53
54	15.6	98.018	3494.708	54
55	15.8	99.274	3593.982	55
56	16	100.531	3694.513	56
57	16.2	101.788	3796.301	57
58	16.4	103.044	3899.345	58
59	16.6	104.301	4003.646	59
60	16.8	105.558	4109.203	60
61	17	106.814	4216.017	61
62	17.2	108.071	4324.088	62
63	17.4	109.327	4433.416	63
64	17.6	110.584	4544.000	64
65	17.8	111.841	4655.840	65
66	18	113.097	4768.938	- 25



6

Results from a Comparative Empirical Study in England and Germany on the Learning of Mathematics in Context

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1. SUMMARY

Results of comparative empirical investigations into different English and German educational approaches to the teaching and learning of mathematics in context are presented. These investigations consist of various case-studies carried out with students aged 14-16 from the higher stream of an English comprehensive school or the higher type of German secondary schools. The comparative case-studies dealing mathematically with the surface and volume of geometrical solids or trigonometric function studies evaluate the influence of the different teaching approaches on the students' image of mathematics, their comprehension of mathematical concepts and methods, and their abilities to use mathematics in order to solve real world problems.

2. EDUCATIONAL FRAMEWORK AND METHODOLOGY

A starting point for the project has been the different didactical approaches to the teaching and learning of mathematics in context developed in the last few years within the international debate. Ideally it is possible to distinguish two different **schools of thought** in the international debate:

i) a so-called **pragmatic school of thought**, which puts emphasis upon utilitarian or pragmatic goals, namely the ability of the students to use mathematics for the solution of real world problems

ii) a so-called **scientific school of thought**, which places formal, science oriented goals in the foreground and emphasises the ability of the students to establish relations between mathematics and the real world.

In practice these schools of thought will not be found clearly separated, rather they are embedded into different educational systems and are represented to a different extent by mathematics educators or teaching materials. The Anglo-Saxon discussion may be described as more pragmatically oriented, originally this held more for the North American discussion. In England such approaches gained more importance in the eighties around the discussion of the Cockcroft Report. In the last few years the educational debate has changed towards an emphasis on general problem-solving skills (so-called strategic skills). The German debate may be characterised as scientifically oriented for the higher ability students (in the Gymnasium) with relations to the pragmatical school of thought for the intermediate and lower ability students (three different schools of thought can be discriminated within the German debate, for a description see Kaiser-Messmer, 1991).

So far **no comparative empirical studies** on the **effects** of these different schools of thought on **teaching-and-learning-processes** have been carried out, although comparative studies are appropriate to point out the strengths and weaknesses of the different approaches. There exists only various quantitative studies (e.g. SIMS) which are based on multiple-choice tests and do not consider either classroom processes nor the teaching and learning of mathematics in context.

We therefore decided in a collaborative project of the universities of Kassel and Exeter to carry out an **own study** restricted to the English and (west) German school system, which is based on observations in the classroom and focusing on applications and modelling. Comparative empirical studies have to take into account the differences between the school systems as well as the differences in the underlying educational philosophies which influence the educational systems. Concerning the English and the German school system **significant differences** exist, which may be characterised by the following catch-phrases:

a) comprehensive school system in England vs three track selective school system in Germany

b) early specialisation in England vs emphasis on broad general education in Germany

c) (still) flexible curriculum in England vs compulsory nature of the curriculum and compulsory core subjects in Germany.

Furthermore, different **educational philosophies** are dominant in England and Germany, which influence significantly the prevailing teaching approaches and teaching styles, materials used and so on. Holmes and McLean (1989 and 1990) develop a description of European school knowledge traditions, in which they characterise the educational philosophies dominant in England and Germany as follows:

i) The **English school knowledge tradition** is described as **humanistic**, based on the principles **morality** (ideal of the Christian gentleman), **individualism**, and **specialism**. The classical roots of the English school system have led in their point of view to the élite orientation as an important characteristic of the English system, furthermore to the separation of education and training, the latter not important in the general educational system. They further argue that mathematics teaching has little concern for the formal learning of principles and insight into general mathematical rules, but is based on the assumption that children grasp general principles as a result of active discovery and active work through a series of examples. This had led to the wide use of example-based, individualised work in English mathematics teaching.

ii) The **German school knowledge tradition** is characterised as **encyclopedic** as well as **naturalist**. The encyclopedic tradition has led to a high relevance of the principles of rationality and universality for the élite education and subsequently a high importance of general education. The naturalistic view aims to connect school life with the community and environment and has led to work-oriented approaches for the mass education. For mathematics teaching this educational philosophy emphasises the understanding of structures and general principles and leads to a low importance being attached to active work through examples. In this holistic view of knowledge, the understanding of structures is seen to be more important than deep knowledge in single areas.

This is the background of our comparative project, in which we are seeking to make comparisons between the German and the English approaches to teaching mathematics and its applications. In detail we want to evaluate questions like the following:

a) Which approach is more appropriate for promoting the ability to **apply mathematics in real-world examples** and provides an adequate background for tackling real-world problems?

b) Which approach is more appropriate for developing a **balanced**

image of mathematics as a science or a comprehensive understanding of the mathematical concepts and methods used?

We developed research-guiding hypotheses on these issues, amongst others we supposed strengths of the English approach concerning the promotion of applied problem-solving abilities and strengths of the German approach in the area of insight into the underlying mathematical structure and conceptual comprehension. We have based our project so far on case-studies, which are thought to be especially suitable for generating research-guiding hypotheses and therefore match the exploratory character of the project. Till now, several case-studies in English and German school classes have been carried out based on teaching materials which are characteristic either of the pragmatic or the scientific school of thought. The case-studies have been limited to the higher stream of (English or German) comprehensive schools or the (German) higher school type at the lower secondary level (Gymnasium). As a research method the techniques of participating classroom observations, attitude and achievement tests have been used.

3. DESCRIPTION OF THE TEACHING OBSERVED

Two extensive case-studies have been carried out so so far, the first dealt mathematically with **surface and volume of geometrical solids**, the second dealt with **trigonometric functions**. Due to space limitations I will restrict myself on the first study (for a more detailed description in German see Kaiser-Messmer/Blum, 1993, for a description in English see Burghes *et al.*, 1992). In this study **two English groups** of year 10 in the English school system (age 14-15) of the higher stream of a comprehensive school in a commuter town near London and **three German groups** of year 10 in the German school system (age 15-16) of the higher type of German secondary schools in a bigger German town participated. The two English groups comprised 28 students, of that 14 girls, and 27 students, of that 11 girls. Two of the three German groups were from the the same school with 24 students, of that 16 girls, and 14 students with 3 girls. The group from another school comprised 16 students with 9 girls.

The **lessons in England** were based on the "**Enterprising Mathematics Course**" from the Centre for Innovation in Mathematics Teaching (Exeter), which is characterised by its strong emphasis on the development of mathematical concepts out of real-world contexts and its individualised approach (see Hobbs/Burghes, 1989). The teaching unit observed was the unit "**Containers for Everything**", which is structured as follows:

i) classification of different containers according to their geometrical shape as starting point

ii) models of pyramids and combination of three pyramids to a prism or cube, calculation of volume

iii) volume of prisms and cylinders

iv) models of cones, curved surface area and volume of cones

v) volume and surface area of spheres.

The examples were taken from everyday life. They dealt especially with the design of packaging and the comparison of different containers.

Similar teaching approaches could be observed in the two groups, who spent 5 weeks on the unit; in detail: the students worked 3 weeks on the **new topics** developed in the teaching materials, afterwards the groups spent 2 weeks on their **course work** on related themes (like minimal surface area of a juice container with volume given using cubes, prisms ...). Individual work or work in pairs, with teacher help when necessary, was the dominant interaction form.

The **lessons in Germany** were based on the textbook '**Mathematik heute**', which uses many real-world examples from everyday life, architecture and technology, and is characterised by its carefully graded concept introduction. The teaching unit was structured as follows:

a) slant drawings, surface area and volume of prisms and cylinders

b) surface area and volume of pyramids and cones (based on the theorem of Cavalieri)

c) surface area and volume of spheres.

Different teaching approaches could be observed in the 3 participating groups: The teacher in the first group with weak students, who had only 3 weeks teaching time, put more emphasis on **explanations** of the underlying geometrical relations and gave low value on the performance of algorithms. In the 2 other groups with 6 weeks teaching time, the teacher highly emphasised **precise concept introduction** as well as **accurate realisation of the algorithms**.

In the first group, class discussion structured by the teacher was the dominating teaching form, in contrast to the two other groups, which were taught in a **strongly guided class discussion** centred on the teacher with only short exercise phases. During the classroom observations each group participated in 1 or 2 teacher-developed classroom tests on the themes.

4. RESULTS OF THE STUDY

4.1 Teaching Styles and Teaching Content

The classroom observations showed that there were remarkable differences concerning teaching styles and teaching content between the mathematics teaching in England and Germany. The main aspects were the following:

* In German mathematics teaching great store was set by the **class discussion** between teacher and all students as a group, whereas in English mathematics teaching **individualised work** was the dominating teaching-and-learning-form.

* The German teachers **dictated** more **strongly** the **interaction** and pace in class than mathematics teachers in English schools, which allows one to cover more topics in the same time in mathematics lessons win Germany than in England. In English classes observed, teachers often had difficulties in adequately directing the individualised work of all the students.

* A main feature of mathematics lessons in Germany was its overall **orientation towards tasks**, for which nothing similar in English mathematics lessons could be observed.

* In mathematics teaching in German schools **short problems** with definite solutions dominated against **comprehensive problems** tackled more often in English schools.

* English mathematics lessons more often dealt with **real-world examples** than German mathematics lessons and they were very often practical and investigational, whereas in German lessons structured, less-practical examples dominated.

* In German mathematics teaching great store was set by **precise and correct mathematical speech** and writing, whereas in English mathematics lessons it was often regarded as old-fashioned to pay particular attention to precise and correct formulation.

* The German mathematics teaching was characterised besides its conceptual precision, by its orientation towards a **mathematical structure**. On the whole it was common for a large area of mathematics to be taught in one go in Germany, whereas in England it was usual for the subject matter to be delivered through the year in **small parcels**, which as a consequence built up difficulties for the overall understanding of the topic.

4.2 Attitudes Towards Mathematics

At the end of the teaching units observed, an attitude test (in connection with an achievement test, see 3.3) was carried out with the students, which points out that the different teaching approaches have led to remarkable differences between the English and the German students in their **attitudes** towards and their **image of mathematics**. The main issues are briefly: Among others, German students emphasised aspects of mathematics as a science such as the **logical structure of mathematics** or the theory-orientation, whereas more English students named aspects concerning the **teaching-learning process.**

In the following a few typical examples of students' answers to the question on whether mathematics is different from other subjects:

Yes the maths lessons are different to other lessons as most of the time, we are allowed to work in pairs or groups and so it makes solving problems easier as the task is shared. The differences lies in my opinion in the logical structure. (translated by G.K-M.)

No differences between English and German students emerged concerning the subscribed relevance of mathematics for life. Many English as well as German students described mathematics as **highly important for life**, but most of the students were not able to give substantial examples in which they really have used mathematics in daily life or other subjects. They only were able to name areas like shopping or banking. A few typical examples:

Maths is very relevant to everyday life, it teaches you how to cope with different situations and problems e.g. adding up bills, choosing my bank account, etc.

I counted the amount of calories in my daily intake.

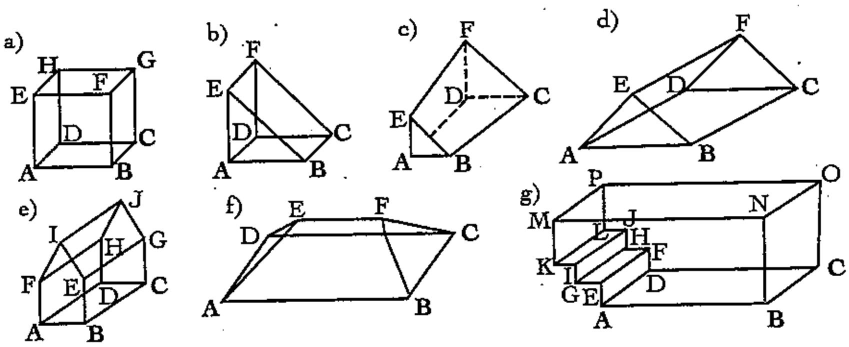
I have learnt to check my change. (translated by G.K.-M.)

4.3 Achievement in Real-world Examples and Concept Comprehension

The concept comprehension and the achievements in tackling real-world examples were tested with the participating English and German groups about 6 weeks after the end of the teaching units. Because of the small sample size I will emphasise the interpretation of the results on the basis of the teaching observed in contrast to pure number comparisons. 51 English and 41 German students participated in the test, which covered in 13 questions with 31 parts, questions on concept comprehension, usage of standard methods to solve real-world examples, and mathematising problems within the mathematical context of prisms, pyramids, cones, cylinders and spheres. The test time was 45 to 50 minutes. 44% to 48% of the tasks were tackled by the German resp. the English students. Due to space limitations I will present only a few of the results.

Prism

B1. Which of these solid shapes are prisms? Mark in the picture one cross-section.



This question should examine how far the students have gained an appropriate concept **comprehension**.

A few more German than English students have given correct solutions, namely 34% compared to 24%. A detailed analysis of the results shows that the two German groups with the more able students have nearly completely failed in contrast to the weaker group.

A *possible explanation* might be the following: In these two groups the focus had been within the concept introduction phase on the classification of real objects. It seems plausible that the students did not sufficiently integrate the action and the representation level of the mathematical concepts introduced, in contrast to the third German group and the two English groups which had learned to read graphical representations of real objects.

Change of Cylinder Volume

B5. How does the volume of a cylinder change, when

- a) the height is (i) doubled (ii) trebled
- b) the radius is (i) doubled (ii) trebled
- c) the height and the radius is (i) doubled (ii) trebled

Give a general rule to describe the changes.

This task should test how far the students had developed the ability of **functional thinking** (i.e. 'What happens if ...').

Compared to the results of the foregoing task a reverse picture arises: Significantly more English than German students have given correct answers, e.g. 49% resp 28% of the students have correctly solved the whole task, only 12% of the German students compared to 39% of the English students solved part c) correctly. Furthermore the German students were strongly oriented towards rules.

Compared to the results of the foregoing task, these results might be surprising, but taking into account the classroom observations, reasonable **explanations** are possible: The English mathematics lessons had dealt with similiar tasks quite often, whereas such problems had been treated in the German lessons very seldom. This confirms an observation, well-known in mathematics education, that functional thinking cannot simply be expected as a transfer after discussing the underlying functional relations, but has to be trained explicitly.

Pyramids

B6. The volume of a pyramid equals $\frac{1}{3} \times$ base area $\times$ perpendicular height.

Give reasons for this rule. (Try to remember the explanations which have been given in the topic unit.)

This question should test if the different approaches to substantiate this formula, common in the English and German lessons, had influenced the **remembrance** of the explanations after a few weeks.

Significantly more English than German students have remembered the basic idea that three pyramids, equal in volume, can be combined to a prism or cuboid (43% compared to 20%). Only the most able German students remembered the detailed explanation given in the German lessons based on the theorem of Cavalieri. 73% of the German students compared to 39% of the English students did not even try any solution (reason: 'I cannot remember.').

These results lead to the hypothesis that the approach to explain formulae based on activities of the students, morè common in English than in German mathematics lessons, is more suitable at least for the majority of the students. The mathematically more precise explanations, common in German classrooms, may lead, at least for the majority of the students, to a complete failure in memorising any explanations.

Mathematising Problem

B12. In forestry, a rule of thumb is used to determine the volume of a tree to be cut. This is based on the diameter in the middle of the tree and the height of the tree.

a) Develop such a rule.

b) Why is this rule of thumb not accurate?

c) How would it be possible to find the volume more precisely?

This partly structured problem should test the students' abilities to **mathematise unknown real-world situations.**

Only a few students have tackled this problem, many more German than English students (20% to 5%) with the German students receiving significantly better results. Two German students and one English student developed a correct rule of thumb, 15% of the German and 4% of the English students have given correct reasons why this rule of thumb is not accurate. Only one German student developed a proposal for an improvement of this rule of thumb (mean of the two basic areas × height), two more English resp. German students have given partly right ideas. The small number of answers allows only a few interpretations. It is remarkable that the observed English mathematics teaching with its stronger orientation towards real-world examples did not succeed in enabling the students to mathematise easy real-world problems. This points out that mathematisations are intellectually very ambitious and very difficult to learn for all students, independent from the underlying didactical approach.

It is generally noticed that the students with good achievements in the whole test performed well in the mathematising problems. This leads to the hypothesis of a strong **relation between mathematical and applied problem-solving abilities.**

5. CONCLUDING REMARKS

It can be stated that the reported empirical studies have shown that there exist big differences between the English and the German mathematics lessons concerning the dominant teaching style, teaching content, and relevance of real-world examples. Furthermore, significant differences in the attitudes and achievements between the English and the German students could be seen, but smaller than expected. As described, differences between the English and the German students' image of mathematics could be seen. In contrast to our research guiding hypotheses, described in the beginning, both groups had severe difficulties to solve real world problems with other than standard methods. Furthermore, in contrast to the starting hypotheses as well, the results concerning the concept comprehension were varied with strengths and weaknesses of both groups. Further studies on a broader scale are necessary in order to come to more secure knowledge about the influence of the different teaching approaches, common in English and German mathematics classes, and on the attitudes and achievements of the students. We have therefore started a longitudinal study which aims to examine the development of the mathematical knowledge of larger samples of students in both countries at the end of their compulsory schooling (for a description of the planned study see Burghes et.al., 1993).

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8

Remodeling Mathematics Teachers' Conceptions Using Performance Assessment Activities

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SUMMARY

Workshops were conducted for middle school mathematics teachers to expose them to new approaches to instruction and assessment. Since teachers' conceptions of mathematics, instruction and assessment seem to influence teaching practice, one goal of the workshops was to challenge the teachers to reshape and integrate their views with the new approaches. This chapter will first describe how performance assessment activities are proposed as a way to integrate instruction and assessment for enhancing and assessing higher order thinking in students. The workshops will be described briefly and the results discussed. The workshops included teachers taking on the role of students to work on a problem and then returning to the role of teacher to reflect on that experience. Evidence indicates that the experience of the workshop provided specific opportunities for the teachers to question, rethink, and remodel existing conceptions of mathematics, mathematical modeling, and issues related to instruction and assessment.

1. INTRODUCTION

The current reform in mathematics education calls for a de-emphasis on drill and practice "basic skills" type of activities, and an increasing emphasis on open-ended, real-world problem situations (NCTM, 1989). Real-world problem situations are realistic (not contrived illustrations), create the need for a mathematical model, and allow for multiple solutions and/or solution paths (Burkhardt, 1981; de Lange, 1987; Lesh and Akerstrom, 1982). A mathematical model is a situated mathematical system that contains mathematical objects, operations on those objects, and the relations between them for "describing, explaining, constructing, modifying, manipulating, and predicting our increasingly complex world of experiences" (p. 21, Lesh and Lamon, 1992). The mathematical modeling of real-world problem situations is generally perceived as having the potential to contribute to the development of higher-order thinking, and the assessment of performance and conceptual understanding.

Performance assessment, also referred to as authentic or alternative assessment, is a form of assessment that requires the students to perform a task rather than select an answer to a multiple choice item (Zimmermann, 1992). Performance assessment activities encourage the student to demonstrate their knowledge and understanding. Performance assessment tasks that involve open-ended, real-world problem situations allow for multiple levels and types of solutions. These types of problems have the potential to provide teachers with information about students' higher order thinking about mathematics that is not possible from more traditional forms of assessment.

The performance assessment activities used in the workshops were taken from the PACKETS® *Program: Performance Assessment for Middle School Mathematics* developed at Educational Testing Service, a national educational measurement institution based in Princeton, New Jersey, U. S. A. Each PACKETS® activity includes a newspaper article followed by readiness questions that focus on understanding the context of the article, a model-eliciting problem based on a real-life situation for which the solution must address the needs of a client (i.e. editor for the local newspaper), a model-exploration problem, and a model-application problem. Each problem is designed as a project-sized activity that is worked on by small groups (3-4 people) for about 40-50 minutes, and then solutions are shared and discussed by the whole group. PACKETS® activities were specifically developed as instructional tools for classroom use as a way to integrate instruction and assessment, rather than as test-like items for accountability purposes in large-scale assessment (Katims, Nash, and Tocci, 1993) that perpetuate the separation between instruction and assessment.

The implementation of mathematical modeling in the classroom has been proposed as a way to align instruction with the new directions in mathematics education, especially with respect to the emphasis on using activities that elicit higher order thinking in students. To implement mathematical modeling activities successfully, teachers should have clear ideas about how to mathematize problem situations, the characteristics of appropriate mathematical models, and how to assess them. Teachers' ways of thinking about real-world problem situations, including their conceptions of mathematical models, modeling, and mathematics in general, influence the ways that those problem situations are implemented. Initial information from the teachers who participated in the workshops indicated that they were unclear and had misconceptions about these ideas.

Thompson (1989) argues convincingly that an essential feature of programs that aim to broaden teachers' conceptions of problem solving includes active involvement in problem solving and time to reflect on that experience. To assist teachers in clarifying their ideas and changing their conceptions about mathematical modeling of real-world problems, it seems reasonable to have teachers experience working on such problems (eg. PACKETS® activities) from the students' perspective. This experience can then be used to generate a reflective discussion about mathematical modeling, characteristics of models, implementation of such activities and related assessment issues. Two workshops for middle school teachers, one in Israel and one in the United States, adopted this approach to shaping teachers conceptions.

2. DESCRIPTION OF THE WORKSHOPS AND PARTICIPANTS

Both workshops had similar goals and were conducted in similar ways. The workshops were conducted to identify and shape the conceptions of the participating teachers with respect to mathematics, mathematical modeling, and the use of real-world problem situations in the teaching of mathematics in the middle school. The workshops were designed so that the teachers would have the experience of doing a problem as students, and the opportunity to reflect on this experience both as students and as teachers. The workshops consisted of three stages: Stage 1 involved gathering preliminary information about the teachers' conceptions of the nature of mathematics, real-world problem situations, mathematical models and modeling, through discussion and journal entries; Stage 2 involved small groups of teachers taking on the role of students to read the newspaper article, work on the PACKETS® problem activity, and then present and discuss their solutions; Stage 3 involved returning to the role of teacher to reflect on the whole experience through discussion and journal entries.

In Israel, 124 middle school tutor teachers participated in a sequence of five workshops that focused on issues related to the use of performance assessment activities. About 20-30 teachers participated at one time in each session of the workshop. All the teachers had between 8-15 years of teaching experience and had received extra in-service training in mathematics education and tutoring skills. A tutor teacher spends half a week as an ordinary teacher and the rest of the week tutoring other teachers in other schools in instructional practices, curriculum implementation, and other innovations.

In the United States, 15 middle school teachers came from several school districts throughout a mid-atlantic state to participate in a three week summer institute as a part of their continued participation in the Teacher Enhancement Project (TEP). The TEP is focused on supporting teachers to align their instructional practices with the NCTM Standards (1989). One of the main goals of the project is to encourage teachers to use mathematical modeling of real-world problem situations with their students. Teaching experience ranged from first year teachers to those with more than 20 years of experience. Mathematical backgrounds varied from those with minimal mathematics preparation required for elementary certification to advanced mathematics preparation required for secondary certification.

Both workshops used the same PACKETS® activity. Each teacher received a newspaper article discussing the United States 1990 census information. The article included a "map" of the United States that distorted the size of each state according to the size of its population; for example, California appeared much larger on this "map" than Texas since the population of California is much larger than the population of Texas. An attempt was made to retain as much as possible the relative shape and placement of each state so as to recognise the "map" as depicting the United States of America. The problem situation was to produce a similar map so that a friend could write an article about the census information from the countries in North, Central, and South Americas. The task included constructing the map and including an explanation of how the map was created. Participants were provided with a map of the Americas and a list of the populations for each of the 22 countries. Other materials such as calculators, graph paper, etc. were available upon request.

3. RESULTS

The results from the workshops in Israel and the United States were found to be quite similar and in agreement with their findings. In an effort to avoid redundancy, the results have been combined in the following discussion.

3.1 Results from Stage 1.

Several misconceptions were identified concerning mathematical models and modeling. For example, typical comments from teachers include:

"the most important thing in mathematics is skills and this is what teachers should be accountable for;" and "the models are sometimes an overload" (i.e. something extra that is not essential). Other comments from teachers indicate a general agreement that using a real-world problem situation would take time away from their "real" teaching which involves helping students achieve a certain level of competency in basic skills. These comments indicate a misconception that activities involving higher-order thinking are less important than practicing basic skills.

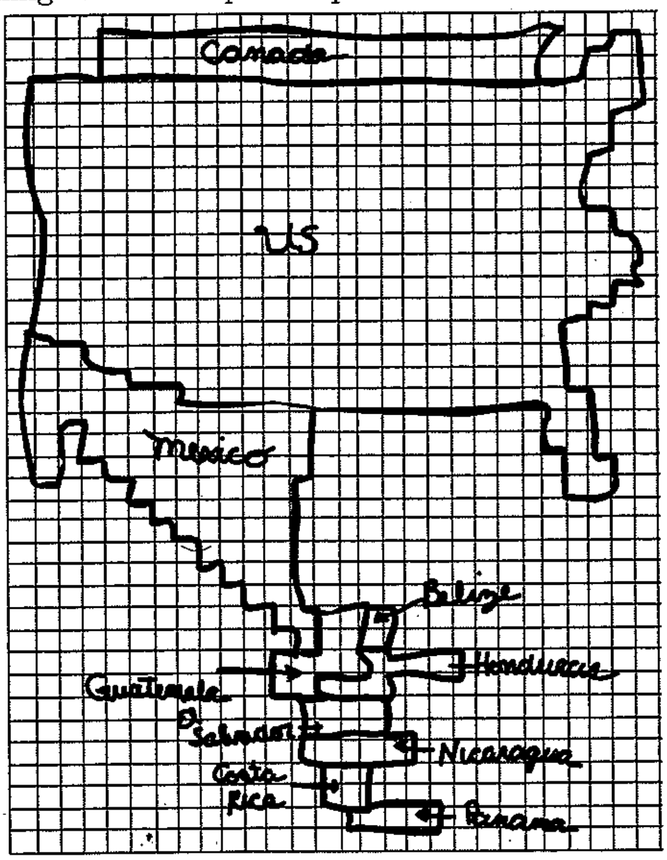
With an emphasis on teaching basic skills through drill and practice, a related misconception is clearly evident concerning the view of mathematics as rules and procedures to be followed in precise ways to arrive at the one correct solution to a problem. In other words, problems in mathematics have one correct solution and solution path. Another misconception that emerged from the teachers' comments was that mathematical models are formulas, (eg. distance equals rate multiplied by the time, $D = R \cdot T$), or they are concrete or visual representations used to teach mathematical concepts (eg. abacus, cuisenaire rods, or the unit circle in trigonometry). In particular, models are used to illustrate or solve specific problems. For example, if a problem is identified as a "distance, rate, and time" problem, then that is the only model that is appropriate to solve that problem.

Another misconception reflected teachers' beliefs that only "great" mathematicians could invent or create a mathematical model; therefore, they also believed that models could not be assessed by teachers but are just "used."

3.2 Results from Stage 2.

As a part of doing the PACKETS® activity, each teacher group constructed a product, presented their product to the whole group, and justified their approach as a solution to the task. The models used in constructing solutions could be identified or classified in a variety of ways depending on the focus of the analysis with respect to the mathematical objects used, the relationships and operations used, or the representations used to "picture" the situation (for a complete discussion and analysis of the teachers' products see Amit and Hillman, in preparation). In this paper, the models were identified by focusing on the mathematical objects that were used such as numerical units, units based on areas, and units based on ratios or percentages. The models are not necessarily mutually exclusive in the sense that some objects appear in more than one model, but they are used in different contexts. At least five different models were identified in the teachers' products. The first four models were generated by teacher groups in both Israel and the United States. The fifth model was created by a teacher group from Israel.

The first model was based on a numerical external unit. Examples included using an external unit such as one square on the graph paper equals one million people, or one square equals one-tenth of a percent of the total population (see fig. 1). The population of the U.S.A. was given as 252.5 million, so that using the first scale requires 252.5 squares on the graph paper, and using the second scale requires about 360 squares since 252.5 is approximately 36% of the total population of North, Central, and South Americas. Some of the products had "rectangular" countries, some products showed a concern for maintaining relative shape and placement.



one square=0.1% of the population

Fig. 1 Example of Model 1 using a numerical arbitrary unit

The second model used a unit based on area. For example, one country was selected as a unit to construct the size of other countries. A teacher group from Israel chose the U.S.A. as the basic unit since it had the largest population, and drawing this largest country on their map first assured them that since all other countries had a smaller population, they would all fit on the page. They constructed other countries by estimating how many times smaller the population of each country was compared to the population of the U.S.A. and reduced the area of each country on the map accordingly. A teacher group from the U. S. A. chose Bolivia (its shape and area as defined on an available authentic map of South America) as the basic unit since its population was close to 1% of the total population. Other countries were constructed by figuring out how many Bolivias it took to make each one; for example, Peru was constructed from piecing together three Bolivias since the population of Peru was 3% of the total population (or three times the population of Bolivia).

The third model was based on a correspondence between the ratios of the population to total population and area to total land area for each country. The total area of the countries on an authentic map was estimated to determine the total "land space" available for the new distorted map. The ratio of each country's population to the total population was calculated and the corresponding ratio of available area on the new map was used for the construction of each country (see fig. 2). A variation of this model by one of the U. S. A. teacher groups considered the graph paper as the total land area available and used the population ratio for each country to determine how much of the paper was needed for each country.

The fourth model involved a similar strategy but used percentages (instead of fractions) as the mathematical objects. The percent of the total population was calculated for each country and then used to determine the percent of available land area that was needed for each country. Another version of this model was created by a teacher group from Israel by using a circle graph for the total available land for the countries, and used the population percentage for a country to determine the size of the "slice of the pie" for that country (see fig. 3).

The fifth model ranked the countries from the largest population to the smallest population and used a bar graph to represent the size of each country with the height of the bar measuring the size of the population.

3.3 Results from Stage 3.

Teachers' comments after the experience of doing the PACKETS® activity provide insight about the misconceptions that were identified in Stage 1. During and after the activity, some of the teachers asked "Where is the mathematics?" To answer this question, the teachers were asked to make a list of the objects they used in their solutions such as units, scales, areas, proportions, fractions, percentages, and estimation. They also generated a list of the operations they used such as comparing, ranking, multiplication and division. These lists convinced the teachers that they were "doing" mathematics. Moreover, it was clear in the discussion that they were making connections between domains of mathematics that are usually perceived and taught as discrete topics such as area and percentages.

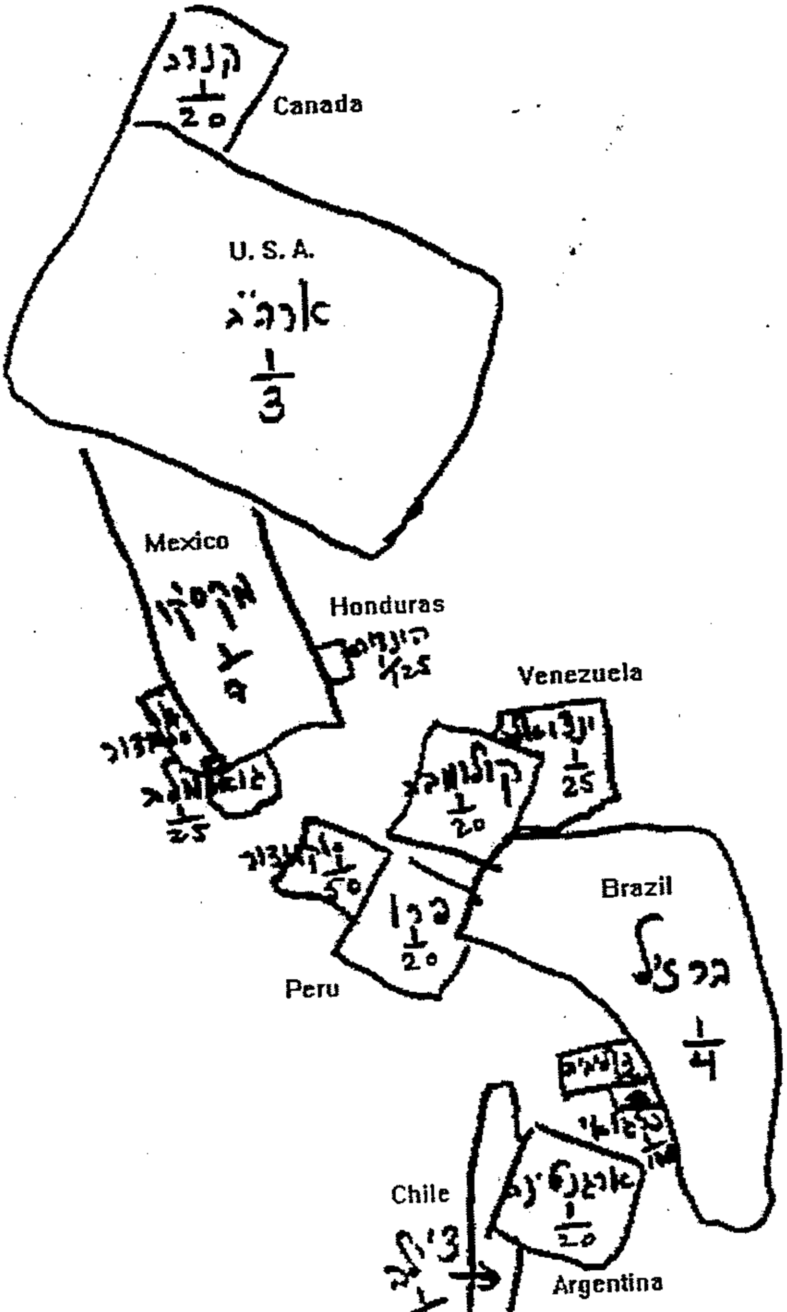


Fig. 2 Example of Model 3 using ratios.
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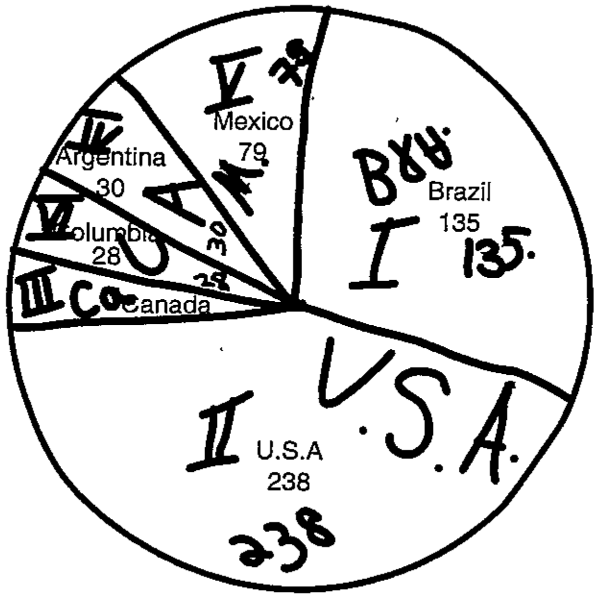

Fig. 3 Variation of Model 4 using percentages.

Each group explained how they mathematized the situation using the terminology of mathematical objects, operations, and relations with the items from the lists. Then the teachers went back to the question "Where is the mathematical model?" The teachers agreed that their solutions contained a "prototype model" that could be used to help construct solutions in other situations. Two examples of other possible situations where this prototype model could be used were given; one teacher suggested creating a map to represent the amount of rainfall in the different districts of Israel, and another teacher suggested creating a map of the original thirteen colonies in North America that represented any one of a number of variables such as population density, or the amount of gross income generated from exports. The teachers were thrilled that they could create a model, and some of them mentioned that they might have created models before but were not aware that was what they had done. The teachers' comments also reflect a concern with evaluation of the models and whether it made sense to talk about a "good" model. Several teachers (in both Israel and the U. S.) argued that some of the products did not qualify as a "map," referring specifically to the bar graph and circle graph solutions. These products, although created in Israel, had been shared and presented as possible solutions to the teachers in the U. S. The ensuing discussion in both workshops revolved around the question "What is a map?" As a result of this discussion, the teachers agreed that to evaluate or assess products constructed as solutions to a task, there is a need to be clear about the expectations or requirements of a task solution. Clear expectations about what is needed for an appropriate solution to the task need not, in fact should not, limit the number of possible models.

The teachers acknowledged that there is more than one model that could be constructed to meet the needs of the client. They seemed to agree that rather than look for a single "correct" answer, there is a need to identify new criteria for the assessment of mathematical models.

4. CONCLUSIONS

To promote the implementation of mathematical modeling and help teachers become aware of how to mathematize problem situations, the characteristics of mathematical models, and ways to assess them, two workshops involving performance assessment activities were conducted with middle school teachers in Israel and the United States. The experience of performing, presenting, and discussing their solution responses provided the opportunity to confront previously identified misconceptions. The misconception concerning the uniqueness of a solution to a problem or unique model was confronted with the production of several significantly different models that reasonably solved the problem situation. The conception that only "great" mathematicians could create a model was confronted with the fact that each one of the teachers contributed to the creation of their group's model. Moreover those models were recognised by the teachers as "prototype" models that described systems containing mathematical objects, operations, and relations which indicated a dramatic change from their initial conception that a model is a formula or concrete representation.

An interesting phenomenon that occurred was the considerable similarity of teachers' misconceptions, construction of mathematical models, and reflective comments on the experience by two different groups of teachers from two different cultures, using different languages, and different settings. We do not claim that the kinds of experiences these teachers had (doing a performance activity and reflecting on that experience) entirely reshaped their conceptions about mathematics, mathematical modeling, and the related issues of assessment, but there is no doubt that some of their conceptions were confronted. By providing experience of the activities from the student's point of view and reflecting on that experience, the teachers were challenged to rethink their conceptions about mathematics, mathematical models, and issues related to instruction and assessment. It was clearly evident from the teachers' comments that many were reconsidering their conceptions in light of their experience with the performance assessment activities. The opportunity to reflect and discuss the experience afforded the teachers a chance to reshape their conceptions. Certainly providing the experience of doing, reflecting, and discussing new approaches to instruction and assessment are a step in the right direction for challenging and reshaping teachers' conceptions of mathematics, instruction, and assessment.

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9

Mathematical Modelling as a Context for Preservice Teacher Education

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Research on mathematics teachers' beliefs indicates many teachers present mathematics as a "cut and dried" subject in which a single correct number is the object of solving problems (Brown *et al.*, 1990; Thompson, 1992). The current reform movement in mathematics education addresses this problem and calls for a shift to more open views of mathematics (NCTM, 1989). To promote this shift, preservice mathematics teachers need experiences that would help them develop a more flexible understanding of mathematics and teaching mathematics. This more flexible understanding includes an ability to recognise and interpret mathematics in a variety of situations and from a variety of perspectives (*cf* Bauersfeldós, 1988, notion of fundamental relativism; Cooney, 1993). Teacher education can promote development in this ability by providing contexts for teachers to improve their propensity to and skill at reflecting on mathematical problems and their own learning and thought processes. Modelling activities are important contexts for these experiences. The activities described in this paper are from a secondary mathematics teacher education course taken by university students in the fourth and final year of their preservice secondary mathematics teacher education programme.* As part of the course, these preservice teachers participated in data collection and

* The development of the activities described in this paper was funded by the NSF project Integrating Mathematics Pedagogy and Content in Pre-service Teacher Education, TPE-9050016, Thomas J. Cooney, Director.

analysis activities that lead to modelling relationships and eventually constructing a deeper understanding of functions.

In keeping with the concerns addressed above and the NCTM *Standards* (1989), several mathematical content and pedagogical goals provide the general foundation for the course (Shealy and Cooney, 1992). In addressing mathematical content, we wanted the preservice teachers to develop a more informal, intuitive understanding of mathematics, to be able to construct and organise mathematical ideas, and to generate and evaluate mathematical interpretations. We wanted the teachers to be able to connect the mathematics to real-world and applied situations–including connections between different areas of mathematics–and appreciate the historical development of mathematics and the role of mathematics in society. Our pedagogical goals include encouraging the teachers to become active and autonomous learners, promoting interaction and cooperative work among the teachers, placing the teachers in new learning situations, and promoting reflection on their experiences.

1. MODELLING PROCESS

Experiences with mathematical modelling of situations are important to addressing the goals described above. DÁmbrosio (1989) said that the modelling process is the essence of creative, intelligent activity and gives a broad definition: Modelling is the process one goes through in which "one is faced with a situation in a real context subject to an undefinable number of parameters, some of them even unidentifiable" (p. 23). Most often the modelling process is outlined as a cycle with as few as three steps (Lambert *et al.*, 1989) or as many as ten (Niss, 1989). Fig. 1 illustrates a summary model we used in designing our activities (de Lange, 1987; NCTM, 1989; Swetz and Hartzler, 1991).

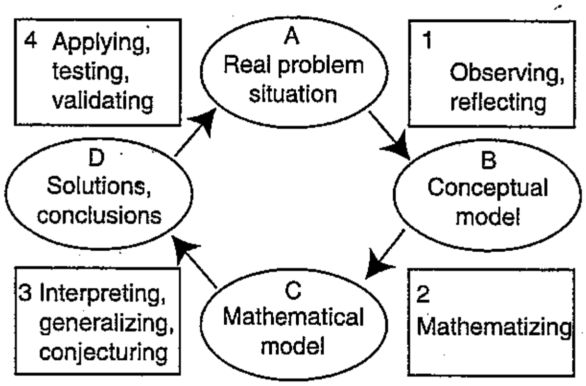


Fig. 1 A model of the modelling process.

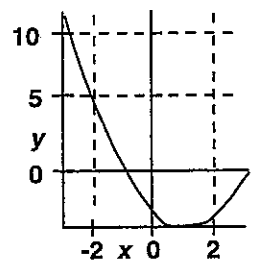
The step from the real world to the conceptual problem–a simplified version of the real world–(A to B) and the final stage (C to D) are of particular importance for impacting the teachers' beliefs about mathematics. Lambert *et al.* (1989), emphasize the importance of viewing the first step from a cognitive psychology perspective. They say in order for the student to make the jump from the real-world to a conceptual model, he or she must have a strong understanding of the real world problem domain and the mathematical domain related to it. Further, their beliefs about the situation will affect how they bring these two domains together to formulate a conceptual model of the situation and how they develop the mathematical model.

Skovsmose (1989; 1990) emphasizes the importance of the final stage (C to D) with what he calls "reflective knowledge." In modelling, Skovsmose says that we need knowledge about mathematics, technological knowledge about the modelling process, and reflective knowledge–"general conceptual framework, or metaknowledge, for discussing the nature of models and the criteria used in their constructions, applications, and evaluations" (p. 767). In particular, he stresses the importance of evaluating our models and conclusions in light of our pre-understanding of the situation and presuppositions, guiding interests that may be present, and being cautious about creating and applying nonexistent objects.

In order to impact teachers' beliefs we placed a great emphasis on these two stages. This emphasis led to the incorporation of three important concepts into the activities. First, the importance of developing and understanding one's mental or conceptual model of the situation (Lambert *et al.*, 1989). Second, considering and analysing one's pre-understanding, goals and interests, and the implications of the model–Skovsmose's (1990) reflective knowledge. Also, in choosing situations, we included those that would stimulate concerns beyond mathematics and challenge the teacher to consider implications of mathematical descriptions, thus developing critical attitudes (de Lange, 1987). Thus, the teachers develop and analyse a mental model, consider data in light of this intuitive model, formulate and interpret a mathematical model, and reconsider the original real-world situation to validate the model. Finally, the participants evaluate the entire activity identifying their goals and biases and evaluating their own thought processes.

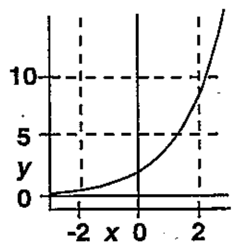
2. MODELLING ACTIVITIES

To help the teachers begin to recognise mathematical situations and practice developing conceptual models, we have the teachers consider the use of dependent-independent and correlational relationships in everyday language and in the media. At first, we provide vignettes (in one vignette, for example, a doctor discusses the effect of a persons cholesterol level on their risk of heart disease) and subsequently we have the teachers find their own examples of relationships that may be described mathematically. The teachers describe the relationships, draw graphs, and compare the relationships to familiar functions as possible models. Increasing or decreasing, rate of change in increase, limits on domains, asymptotic behavior are among the characteristics of relationships the teachers describe intuitively based on these vignettes. Next the teachers participate in several activities in which they investigate characteristics of various functions. The teachers are generally familiar with more formal mathematical development of families of functions–linear, quadratic, higher-order polynomial, logarithmic, exponential, rational, algebraic irrational, and trigonometric. To encourage the teachers to develop more flexible understandings of these families we build on the intuitive base developed in the vignettes. In one of the activities, the teachers classify cards exhibiting functional relationships represented as equations, tables of data, verbal descriptions, and graphs (see Fig. 2, for example). These card sort activities help the teachers focus on characteristics of the functional relationships, usually moving from superficial comparisons to more sophisticated ideas.



The 1990 census shows that Central City has a population of 40,000 people. Recent studies indicate that over the past 20 years the population has steadily increased at an annual rate of 2%. Social Scientists predict Central City will experience this growth rate over the next 20 years. Ms. Callahan has been asked to predict Central City's population for each of the next 15 years.

x	–2	–1	0	1	2
y	1	0	1	4	9



Fred is considering which size pizza is the better buy. He wonders what happens to the area of the circular pizza when the diameter of the pizza is changed.

x	–2	–1	1	2	4
y	1	2	-2	-1	$-\frac{1}{4}$

Consider the six representations of functions given above. Group them into either two or three piles using whatever criteria you wish. How many different groupings did you identify and what criteria did you use?

Fig. 2 Sample card sorting activity

To build on this intuitive understanding of functional relationships, the students participate in several data collection and analysis activities. In one activity, the teachers investigate the relationship between the diameter of the area that can be seen on a wall through a paper tube and, first, the distance from the wall, second, the length of the tube, and third, the diameter of the tube. A second investigation compares the period of a pendulum to the mass, length, and initial displacement angle of the pendulum. The teachers first describe and discuss the characteristics they expect of the relationships–a mental experiment. After collecting the data, the teachers give written descriptions of the relationships, provide tables, and sketch graphs. The teachers then describe the relationships and compare their results to their expectations.

After the data collection activities, the teachers analyse United States census data, investigate problems involving compounded interest, and discuss possible connections between a new business' advertising expenditures and their initial sales. By the end of the investigations the teachers have seen real-world relationships modelled by linear, quadratic, exponential, logarithmic, periodic, rational $\left(y = \frac{k}{x}\right)$ and algebraic $\left(y = \frac{k}{x}\right)$ functions. We then have the teachers compare and contrast the intuitive relationships and the explicit types of functions to strengthen their understanding of the function characteristics.

3. DISCUSSION

Davis (1991), quoting Berlinski, said that "mathematical descriptions tend to drive out all others" (p. 4). Davis followed this quote by saying that "once a mathematical description is in place it is harder to change than moving a grave yard" (p. 4). Research in mathematics teachers' beliefs (Thompson, 1992) provides evidence that teachers have preconceived mathematical ideas that are resistance to change. This difficulty was evident as we worked with the teachers. Consider what happened in one of the investigations (recall that these teachers have a strong university mathematics background). When the students were analysing and discussing United States population data and trying to determine types of functions that would model the data. All the students expected an exponential function to provide the best model. The data from 1790 to 1860 demonstrates almost pure exponential growth to reinforce this idea–consistent 32% growth rate per decade. It happens that from 1940 to 1990, a linear function or a logarithmic function is as good or better than an exponential model (see Fig. 3). In the discussion that followed, one student strongly argued that any non-exponential trend would not continue and that over the long run we would see a continuation of exponential growth. In her words, "Population grows exponentially, that's the way it is." This commitment exemplifies the perseverance of mental models that do not match relationships well and the strength of commitment to prior convictions in light of evidence to the contrary. It is interesting that the teachers who clung to an exponential model of population growth regained their exposure when presented with an alternative model–the logistics curve. These teachers much prefer an established mathematical description to dealing with uncertainty.

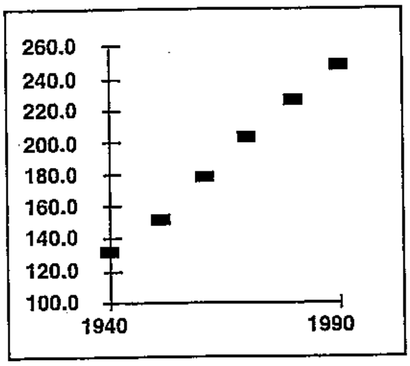


Fig. 3 United States population, 1940-1990

Another problem that arose over the course of the investigations was the tendency for the teachers to give superficial descriptions and expect most relationships to be linear. In the card sorting activities (Fig. 2, for example), first attempts at classification tended to focus on the obvious categories of representation (i.e., graph, equation, table, verbal) and then a linear/nonlinear dichotomy. In the data collection activities, the teachers in every case initially chose to describe the relationships as linear. Subsequent investigation and guided reflection on the activities were necessary for the teachers to begin to look for and distinguish between various nonlinear relationships. The discussion around the relationship between the length of a paper tube and the diameter of the area on a wall seen through the tube provides a nice example of this reflection. The teachers want to make the relationship linear. Mentally extending the experiment, they discuss what would happen if the tube were extended indefinitely. A decreasing linear relationship would imply the diameter is eventually zero. The idea that the diameter approaches zero without reaching zero allows them to construct an understanding of a function that is decreasing but not linear.

Some of the teachers questioned the value of bringing "non-mathematical" ideas into the mathematics classroom (e.g., issues surrounding the uses of the United States census). They were concerned that using data related to such areas as war and disease might offend or upset some students. Considering implications of mathematical models, however, was very important to the teachers becoming more reflective in their learning and teaching experiences and challenging mathematical descriptions. Discussions on the census activities centered around uses of the census, such controversies as undercounting, and the implications of assuming exponential growth when the data from approximately 25% of the history of the United States indicates otherwise. These issues follow the ideas of developing critical attitudes (de Lange, 1987) and reflective knowledge in modelling (Skovsmose, 1990).

Overall the teachers were positive towards the activities and felt that they grew in their understanding of mathematics and teaching. Our subsequent research provides evidence that the teachers involved in these modelling activities have become more open and flexible in their understanding of mathematics. They also plan to use modelling activities in their classroom as they begin teaching (Shealy *et al.*, 1993). Thus, modelling activities, particularly emphasizing the reflective nature of developing conceptual models (Lambert *et al.*, 1989) and, in the last stage, reconsidering the just-completed process and developed interpretation (Skovsmose, 1990), seem to provide the opportunities teachers need to develop the more flexible and powerful understandings of mathematics and teaching mathematics needed to support current mathematics education reform (NCTM, 1989).

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Section C
Assessment

10

Assessment in Context for Mathematical Modelling

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SUMMARY

Credible assessment schemes measure evidence of student achievement, as individuals or within a group over a wide range of activities. This paper shows that item response modelling can be used to develop rating scales for mathematical modelling. It draws on examples and experience from the UK Assessment Research Group, which has proposed comprehensive rating scales to assess mathematical projects and investigations, from the development in Australia both of adult literacy and numeracy scales and of assessment scales for architecture, mechanical engineering and music.

1. HOW SHOULD MATHEMATICAL MODELLING BE ASSESSED?

Ten years have passed since ICTMA1 in Exeter, UK during which time there has been a substantial increase in mathematical modelling activity across all sectors. Modelling has been promoted to develop the knowledge, skills and understanding in mathematics education within a creative environment in which mathematics is applied (usually) to real life situations. These strategies provided the means to increase the motivation and the application of pupils and students from diverse backgrounds and across all sectors. In the course of introducing modelling to the curriculum and from the resulting experience and practice it is clear that the benefits of its introduction extend beyond the bounds initially envisaged by the original aims. Of course our engineering colleagues could have told us that students would learn about project management, social scientists and psychologists would say that personal development and group interaction would result from modelling activities. The symbolic logic in the structure of mathematics in one sense extends communication skills within a rigorous framework but its nature and its use has often served to inhibit communication, and therefore understanding, by undervaluing a students readiness to make naive mathematical statements (Haines, 1980). Linguists would recognise the importance of written and oral communication skills as they are developed in modelling activities as a means to build the complex structures of mathematics.

Mathematical modelling in the curriculum forms part of the wider field of teaching and learning in mathematics for which Burton (1992, p. 5) describes the two philosophical paradigms which dominate the scene. A person holding the Absolutist Paradigm "believes in the value of specified knowledge, which is accorded objective non-negotiable status...This transmission mode of teaching leads inexorably towards the kinds of testing and examinations which... focus on content and skills and their reproducibility" of which the unseen timed written end of course examination is one example. The outcomes from such a restricted assessment are more predictable in terms of reliable judge behaviour than using other tasks which allow for a multitude of student responses.

Table 1 places judge behaviour on an arbitrary scale to represent the view that certain assessment tasks taken in isolation produce consistent indicators with respect to agreement between judges. Multiple choice and traditional closed book examinations are held by the absolutist to be effective measures of student achievement but they do not allow for the comprehensive assessment of the full range of student achievement and convey an inappropriate restrictive view of performances which have value in mathematics. Other modes of assessment do perform just as well, for example, teacher observation checklists can be an effective and reliable measure (Table 1).

Burton continues, "an alternative approach is constructivist and sees knowledge as relative both individually and societally. Far from believing in objective truths, the constructivist assumes that each person's view of their world is individual", but that it is imperative that the individual can relate to the consensual domain of the social environment. This "style of learning relies upon learner responsibility, group communication and the negotiation of meaning between and with learners. It leads to a form of assessment which is reflective, that is self-questioning, rather than performance based. To be effective in the widest possible sense, the evaluation and assessment strategies must be those which are consistent with the social constructivist paradigm." (Burton, 1992, pp. 5-6).

The stark choices between the absolutist and the constructivist approach, reflect the position of mathematical modelling within traditional mathematics courses in higher education. In order that students may receive full credit for their achievements the assessment scheme must be carefully integrated with the syllabus and the pedagogic base. The problems in achieving this integral approach should not be underestimated and were to some extent foreseen by Niss (1993, pp.50-51) in expressing the view "that most current assessment modes are inappropriate to assess high level and complex mathematical activities including applications and modelling. Appropriate modes of assessment for such activities do exist, but they are incompatible with certain traditional requirements of assessment in mathematics." Niss offers no solution to resolve this incompatibility. A more positive approach is taken by Burton (1992, p. 6) who proposes rules of thumb to help in constructing or evaluating an assessment style. The list is by no means exhaustive but it contains certain "truths" that should be crystal clear to those who do not accept the social constructivist approach, rather leaning towards an absolutist view:

1. that the assessment is appropriate to what is being assessed;
2. that the assessment enables the learner to demonstrate positive achievement and to use their strengths;
3. that the criteria for successful performance are clear to all concerned;
4. that the assessment is appropriate to all those being assessed;
5. that the style of assessment blends with the learning pattern so that it contributes to it.

In putting forward a comparable list of eight criteria for creating and appraising more extended assessment practices, Eisner (1993, p. 231) notes that views of assessment vary, stating that "if one wants to identify winners in a race, it makes sense to have runners start at the same place, and if this is not possible (and it is not in academic contexts) they should at least run on the same track." But to identify ways in which students come to interpret, apply and incorporate new knowledge into their current framework requires assessment strategies which allow these features to be recognised and documented. The need for valid and dependable measures for undergraduate teaching at universities is well known; in the United Kingdom, internal and external reviews provided many examples of the problems with assessment of projects and investigations (Haines, 1992). The motivation to document these diverse and complex patterns of learning and achievement in mathematical modelling is clear, and the rewards, not just for the student, but in terms of quality and accountability are high.

2. WHAT APPROACH IS TAKEN BY THE UK ASSESSMENT RESEARCH GROUP?

When faced with a problem in mathematics, or particularly in mathematical modelling, it is natural to begin with principles and procedures that have been successfully applied in the past. Developing areas of mathematics provide examples of real student achievement which are not amenable to traditional assessments, or where they do not address the issues, process, or complexity that should be part of the assessment. For progress to be made the problem must be recognised as of significance and alternative assessment procedures must be developed. However, assessing complex mathematical tasks crosses subject boundaries, driven by the need to provide accurate and comprehensive profiles of students' learning achievements.

Gathering a group of experts to discuss an issue and to provide advice is a well-known strategy. A consensus may not be reached on all aspects, but a meeting provides an opportunity for clarification of the issues, exchanges of views, and establishing the extent of shared meanings. In disseminating the findings of the panel, many bodies assume that the assessment problem has been solved, but a number of issues remain. For example: To what extent was a consensus reached? Where and how did the experts differ? Did their responses cluster in some way? Were some areas more controversial than others? Do these areas reflect a lack of consensus among experts? Should students be penalized for agreeing with the "wrong" set of experts? Where researchers and students do not share a definition of mathematics it is a barrier to using mathematical knowledge in a variety of contexts (Kouba and McDonald, 1991).

A research programme in United Kingdom universities (Haines, Izard, Berry et al., 1993) has used a workshop approach to identify problems in assessment, to attempt to reach a consensus on possible solutions and to test these solutions to obtain evidence on their applicability and suitability. The method adapts an algorithm developed by Griffin and Forwood (1991) and Izard and Griffin (1991) for reporting adult literacy and numeracy. The procedure, in a mathematical context, involves the generation of descriptors, such as generalisations made and proofs attempted, by the assessors, the use of those descriptors on a series of rating sheets to assess actual work, analysis of the results to identify discrepancies in shared meanings and revision of the descriptors. The procedure (Table 2) followed identifies stages interpreted distinctively by the research group.

Stage 1 Experts are invited to describe what they look for when assessing student work.

Comment Experts from twelve institutions in the United Kingdom established the first workshop of the Assessment Research Group. They looked at criteria and assessment procedures for mathematics projects and investigations (Berry and Haines, 1991). In discussing the rationale for assessment, they described how they recognise quality project work presented by students. They provided indicators of competence for use by those assessing projects and investigations. The indicators are descriptors of student behaviour so that when competent judges see a number of these descriptors, the observations provide evidence of achievement. The group size should represent all likely shades of opinion and have expertise from the complete spectrum of student tasks. It is at this stage that shared meanings are achieved.

Stage 2 The resulting list of descriptors is discussed and modified by the group of experts.

Comment This stage produces refinements and establishes a structure. Working in subgroups, participants identified three groups of descriptors-those concerned with the activity of the investigation, with integration of knowledge and skills to tackle a problem, and with delivery of the report in oral and written form. Breaking the task down focused the discussion on the perceived problems. Later workshops set objectives for shorter projects and investigations resulting in a variation in the groups of descriptors.

Stage 3 An edited version of the list of descriptors is used to assess real projects.

Comment The considered opinion of the participants at the end of the first workshop in 1991 was such that at this stage they had captured the essence of what needed to be done. This is a common misconception in such cases, for there remained the question of evidence to support this feeling and the procedure now examines this issue. Both high and low quality projects must be considered since one purpose of assessments is to distinguish between them and to award consistently higher marks to higher quality work. Experts apply the rating scale based on the descriptors to these real projects, recording their independent ratings. Each has to assess more than one project, and the lists of projects for assessment by each expert have to overlap with those of the other experts. In this way several judges assess each project. This procedure can be used on sections of work, for example to check the appropriateness of the descriptors for oral communication skills (Haines, Izard and Le Masurier, 1993).

Stage 4 Item Response Modelling is used to examine variation and discrepancies.

Comment An objective of the assessment is to differentiate between students/projects where there are differences and to quantify those differences. The descriptors should therefore discriminate: if there is a range of achievement then each descriptor should help to find these differences. The need to recognise differences in examiners/judge behaviour is frequently not understood. Using descriptors on actual student work allows checks on which descriptors work, which overlap or are redundant, and which are controversial. Judge performance, in which some consensus is expected but not perfect agreement, can also be explored at the same time. The analyses uses commonly available software tools. The FACETS program (Linacre, 1990) separates judge behaviour from student performance and also may consider the effectiveness of the descriptors used. The QUEST program (Adams & Khoo, 1992 and Wilson, 1992) provides rating scale analyses showing perspectives of student performance and descriptor effectiveness. In each case results are shown on related linear scales which have the same metric: the student achievement, the judge behaviour and the demands of the descriptor continua.

Stage 5 Review of the descriptors, the performance of the examiners and of the students.

Comment The list of descriptors is discussed in the light of the analyses; examiners are advised of their performance relative to their colleagues who helped develop the descriptors. Students may be advised of the examiners views of their work and the relative difficulty of the demands of each descriptor. Such advice focuses students on the skills and knowledge they are required to demonstrate to achieve well on the assessed tasks.

Stage 6 The list of descriptors is revised and submitted to further trials.

Comment This is part of the continuous process of the development of comprehensive scales. Trials are required to establish the credibility of the scales, to extend their applicability and to assist in the role of staff development.

Stage 7 Feedback and monitoring.

Comment Information (about how the projects really are assessed rather than how they were intended to be assessed) is conveyed to the examiners/teachers/students. Teachers and students are told which targets are easiest and most difficult to attain. The issues of leniency/stringency and shades-of-grey/black-and-white are raised with the examiners. In each case, are changes needed? How are they to be made? If we do "kidmaps" or similar (Adams & Khoo, 1992), how should non-fit be handled?

3. WHAT HAS BEEN ACHIEVED IN THIS FRAMEWORK FOR ASSESSMENT?

The UK Assessment Research Group has constructed descriptors which deal with both the process skills and the ability to integrate knowledge and skills to tackle a problem. These descriptors are the main parameters for what in mathematical modelling is loosely regarded as content.

The descriptors reported in this paper deal with shorter projects and investigational work in modelling, pure mathematics and statistics. They do not distinguish between first year, second year, group or individual projects, nor were they developed for the extensive final year projects of many undergraduate courses, although this was the subject of the 1993 Kells workshop. The modelling (M), pure mathematics (P) and statistics (S) descriptors are the main parameters for content, while those relating to "Communication" deal with the delivery of the project, encompassing the written report, group work and an oral presentation. The descriptors for written work (W) were derived directly from a group workshop (Berry and Haines, 1991) while the development of the oral descriptors (O) are discussed by Haines, Izard and Le Masurier (1993).

With each group of descriptors there is an accompanying commentary which indicates the basis on which the judgment is to be made. In order to use the system a given institution or course may select

M1-M9, W1-W9, O1-O9	for modelling projects
P1-P7, W1-W9, O1-O9	for pure mathematics investigations
S1-S8, W1-W9, O1-O9	for statistics projects

as appropriate and construct their own assessment form. The M, W and O destriptors, while presented separately, are a compact package of assessment tools for assessing mathematical modelling completely. Appendix 1 gives the full rating scale for mathematical modelling (M1-M9) including the commentary. Appendix 2 gives the descriptors for the S,W and O scales omitting the commentaries, the P scale is given below. The commentaries are used in the method to establish shared meanings among the assessors and those being assessed.

The descriptors were developed from the premise that each of the three types of project could be described by about 7 descriptors of common activity, although driven by modelling projects they are identified most effectively by considering the processes involved in a pure mathematics investigation. These processes, which intuitively correspond to the mathematical modelling descriptors (M2-M8) are:

P1. Identifies main objectives of the task
P2. Shows understanding of the problem
P3. Identifies possible variables of interest
P4. Possible relationships between variables, or conjectures, explored
P5. Makes a mathematical formulation of the problem or conjecture
P6. Finds a solution or gives a proof
P7. Reflects on the problem

The mathematical modelling process requires two further key descriptors,

M1. States objectives of the task, and
M9. Validates solution

being specific to mathematical modelling. Special skills were identified in statistics projects, so that P2 is split into

S2. Formulation: simplifying assumptions made, and
S3. Carries out experimentation or surveys,

the remaining six P descriptors map directly to S1 and S4 - S8.

The above prototype rating scales show what can be achieved in stages 1,2,3 and 4 of the Griffin procedure, the remaining stages involve further development, testing and validation as a dynamic process for assessment. Hands-on experiences of using the oral scale, a draft scale for posters and interpreting the results of the associated FACET analysis are discussed in the next two sections.

4. ASSESSMENT OF ORAL PRESENTATIONS AND OF MATHEMATICAL COMPREHENSION AND COMMUNICATION

Workshops directed by: Sylvia Dunthorne (Open University, UK), Ken Houston (University of Ulster, UK), and David Le Masurier (University of (Brighton, UK)

In order to fully understand data obtained from the workshops, it is helpful to review the activities on which they were based. Sixteen workshop participants reviewed the development of the oral descriptors and applied rating scales (Appendix 2) to video recordings of student presentations (Haines, Izard and Le Masurier, 1993). The modelling problem was described together with the assignment given to the students.

The descriptors were applied to the presentation of one student group measuring student achievement both individually and collectively as a group. This first application, a trial run, exposed logistics problems faced by the participants highlighting the need for shared meanings at stage 1 of the Griffin procedure. A second group presentation was then shown and the rating scales applied.

The practicalities of understanding the modelling exercise, interpreting the meanings of the descriptors and the completion of the rating sheets should not be underestimated. Different approaches were used for this purpose, some completing the forms throughout the video presentation while others preferred to wait until the end. There was general agreement that the demands were far too great during the trial run, but some improved their performance during the second presentation, again the importance of the early stages of the Griffin procedure is paramount. Further questions arise on whether the performance of the students as perceived by the judges viewing the video is a valid measure of real student achievement. For example, the technical skills of the video production team and the direction of the camera team will have influenced the judges.

Student achievement can be measured and assessed through a variety of modes which have quite different aims and objectives and which produce diverse outputs. An oral presentation is a quite different matter from written comprehension and communication.

In a modelling activity it is common for students to prepare a written report and in some cases to give an additional oral presentation. At the University of Ulster, following such an activity, an innovative style was introduced in which students were invited to communicate their findings by a poster session. The fourteen participants at this workshop were invited to assess posters using given criteria:

PS1 States the problem
PS2 Outlines the solution
PS3 Reports the results
PS4 Uses bold headings
PS5 Good design layout
PS6 Uses illustrations
PS7 Aesthetically pleasing

Although those assessing were experts in their own field, stage 1 of the Griffin process had not been carried out in devising these draft scales. The judges found the criteria easy to use but did not agree on their suitability. The following analysis shows that It would have been preferable to reach some consensus about the descriptors before they were used to assess student work. There was also a difficulty in that the descriptors themselves were used without an associated commentary such as that attached to the mathematical modelling descriptors (Appendix 1).

The workshop participants also considered examples of comprehension tests, students' responses and their assessment so that the suitability of this mode of assessment could be gauged and the extent to which the particular examples given met the stated objectives. Houston (1994) discusses the background to this method, but the need for good data and experimentation in this area is clear.

5. Analysing And Interpreting The Outcomes

Workshop directed by: John Izard (ACER, Australia) and Chris Haines (City University, UK)

The data provided by the workshops described above were analysed separately by Izard using the FACETS program (Linacre, 1990). The program models judge behaviour from student performance and also allows for consideration of the effectiveness of the descriptors used. The results from the analysis are shown on related linear scales; the student achievement continuum has the same metric as the judge behaviour continuum and the descriptor continuum.

ORAL PRESENTATIONS

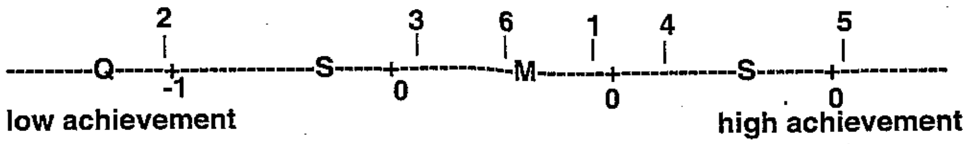

Fig. 1 Six students placed on an achievement continuum for oral presentations. Student 5 had the highest achievement in this sample, student 2 was the lowest. Student 2 was almost two standard deviations lower than the mean.

There were significant differences in the achievement of students in oral presentations judged from the ratings. Interestingly, student 2 was judged to be the worst in this sample, consistent with the results of Haines, Izard and Le Masurier(1993), despite the fact that the judges were novices in this type of assessment and had not been involved in the development process. The ranking for the six students was also comparable. Fig. 1 shows the spread of student achievement ratings from their combined performance on each of the equally weighted oral descriptors.

Analysis of the behaviour of the judges for the oral presentations showed there were significant differences in leniency/stringency between them, they were consistent in their interpretation of the oral scales (reliability 0.79). This is quite surprising, since such a short time was given over to establishing shared meanings between judges. Fig. 2 shows the judges on a linear scale (in logit units), in the same metric as Fig. 1, considered a leniency-stringency dimension.

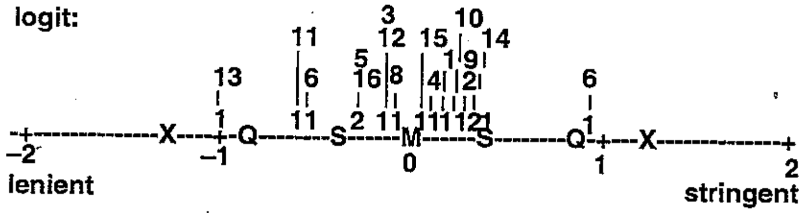

Fig. 2 Sixteen judges placed on a leniency/stringency continuum after assessing oral presentations. Judge 7 is the most stringent; judge 13 is the most lenient; judge 7 is more stringent than any other judge by at least one standard deviation. Judges 3 and 12 mark to the same stringency, as do judges 9 and 2.

Strategies in item response modelling, using student-judge data from rating scales, seek to separate the influences of question complexity from student achievement in the traditional test or examination context. The corresponding emphasis in this type of study may be considered as item demand separated from student achievement. Some criteria are easier to meet than others; knowing which are easier and which are more difficult to satisfy gives us information about the assessment process. The question is "Which descriptors being assessed place the most (least) demands on the candidates?" taking into account difficult issues of ratings (or short-answer marks) presented as 0, 1, 2 or 3 rather than 0 or 1 as for scores on a multiple-choice test question.

Earlier studies (Haines, Izard, and Le Masurier, 1993) showed that the group descriptors (5,6,7,8) were easier to achieve (less demanding) than individual descriptors (1,2,3,4). Here, (Fig. 3), perhaps due to the lack of shared meanings for the descriptors and the use of novices as judges the distinction between the descriptors is not so clear. This finding warrants more investigation, particularly in the light of Eisner's comment (1993, p. 228) that assessment tasks need not be limited to solo performance.

```
                              4
logit:                  3   6|  5  2 1  7   8
                        |   ||  |  | |  |   |
+----------------X--+---Q---------S-------M-------S-------Q--+---X----------------+
-2                  -1                    0                  1                    2
less demanding                                                    more demanding
```

Fig. 3. Oral descriptors placed on a less-demanding/more-demanding continuum. Clearly descriptors 7 and 8, relating to visual aids were perceived by the judges to be more demanding than descriptor 3 on spoken English.

POSTERS

```
logit:
           5             92        4 5  1 6  7   10        3
-------------Q---+--------------S---------+------M--------------S+---------------Q-------+
                 -1                       0                      1                       2
low achievement                                                        high achievement
```

Fig. 4 Ten students placed on an achievement continuum for posters. Student 3 had the highest achievement in this sample, students 9 and 2 were low achievers, one standard deviation lower than the mean.

The students efforts at poster presentations were placed on an achievement continuum for which there was a remarkable confidence in their score, reliability 0.97 (Fig. 4). This high value is partly a function of having a large number of judges, relative to a small number of students. The results of the analysis of the behaviour of the fourteen judges for the poster presentations is displayed in Fig. 5. Participation in the workshops overlapped, so that judges 1,2,4,10 and 13 took part in both the Oral and the Poster exercises. Whilst there was consistency between the judges on this task (reliability 0.81), Fig. 2 and 5 show that the leniency and stringency of an individual judge may vary according to the task being assessed or the occasion on which the task was assessed. Notice that the relative positions of judge 10 and 13 are reversed in Fig. 2 and 5.

Fig. 6 illustrates the demand of the draft descriptors for poster presentations, but does not describe the full story since there were two misfitting ratings both due to redundancy between the descriptors 4 and 7. This highlights the need for careful development of the rating scales making full use of expert panels, trialing and testing through the stages 1 to 5 of the Griffin procedure. In addition the descriptors are clustered about the mean in terms of demand which may point to difficulties in using them to discriminate between students. In constructing tests it is desirable that the range of tasks places variable demands on the student so that both low achievement and high achievement is recognised.

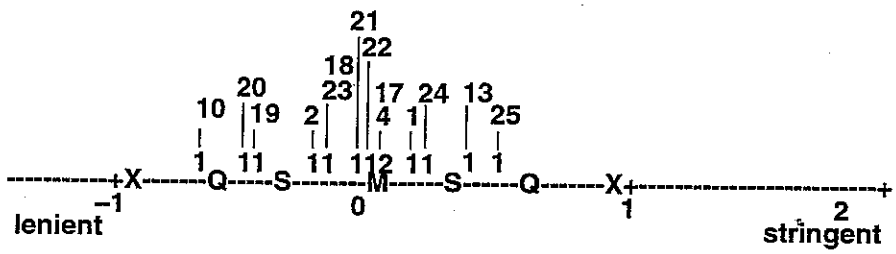

Fig. 5. Fourteen judges placed on a leniency/stringency continuum after assessing poster presentations. Judge 25 is the most stringent; judge 10 is the most lenient. Judges 1,2,4,10 and 13 are also placed on the continuum shown in Fig. 2.

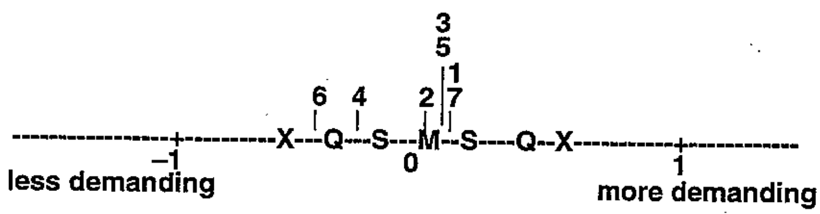

Fig. 6. Poster descriptors placed on a demand continuum.

5. ARE WE ASSESSING COMFORTABLY?

The framework within which the assessment takes place is dynamic, as the use of mathematical modelling has become more widespread so too has the range of student outputs increased. There are exciting new developments in the use of comprehension tests, posters, multimedia activities and experimental laboratories each of which add a dimension to the modelling activity. The experiences of the workshops and of the UK Assessment Research Group as a whole suggest that the methods adopted for assessing mathematical modelling in context are taking us in the right direction. Eisner (1993, p.231) uses an athletics analogy in establishing criteria for assessment, and this analogy can be extended to the way in which a highly trained track athlete performs to a very high standard. During a race if the athlete is feeling too comfortable then there is room for improvement, the competitative edge comes from just being across that threshold into an uncomfortable state. So it is in implementing any of these schemes for assessment, critical self-appraisal while assessing combined with a feeling of being just uncomfortable with what is being done helps to maintain a healthy perspective.

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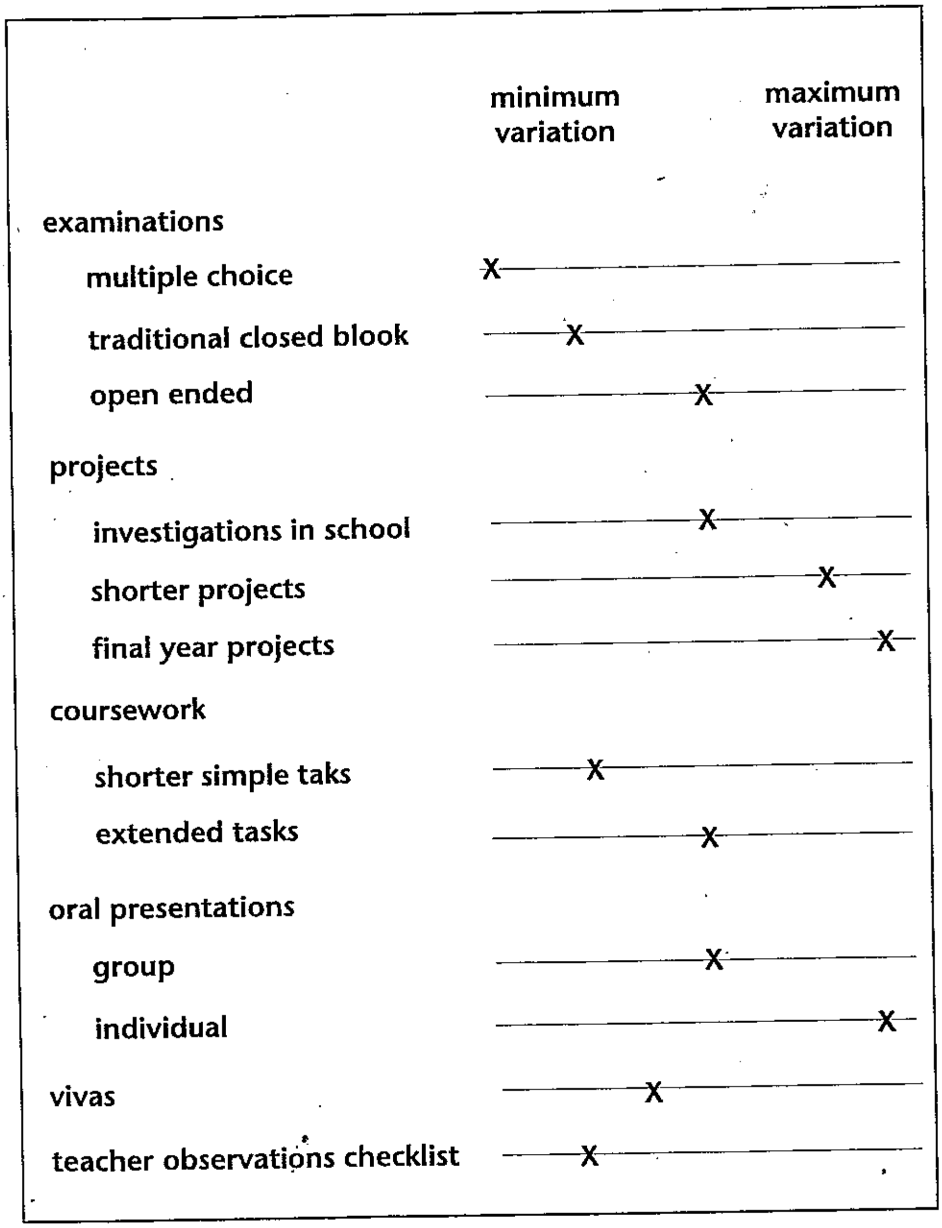

TABLE 1. Typical judge behaviour on a range of common assessment tasks. mathematical modelling activities tend towards a perceived high variation in behaviour, for example in final year projects.

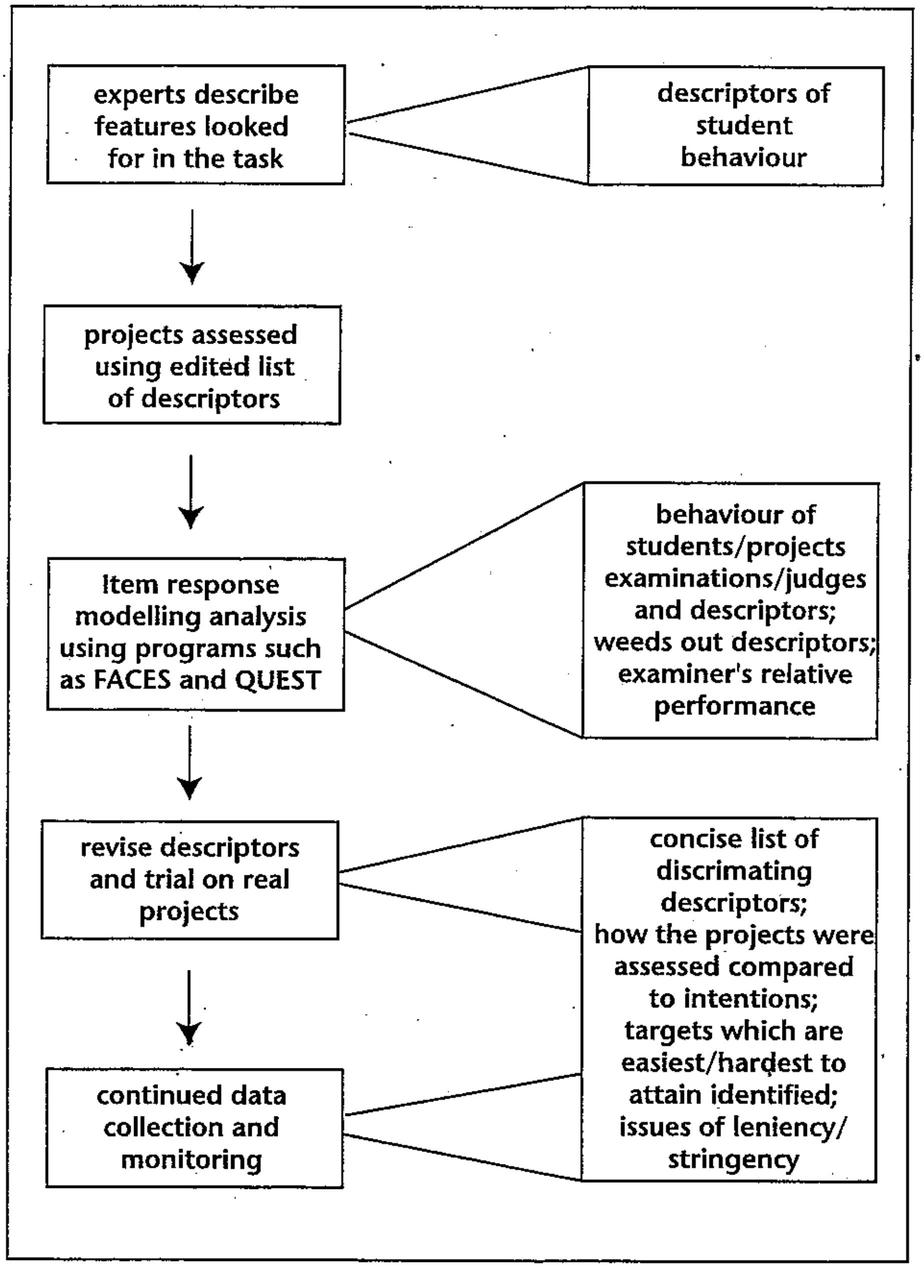

TABLE 2. Methodology for developing sets of descriptors, after Griffin and Forwood (1991), adapted by the UK Assessment Research Group.

APPENDIX 1

MODELLING		high			low	not shown
M1	States objectives of the task	☐	☐	☐	☐	☐
M2	Identifies the main features of the task	☐	☐	☐	☐	☐
M3	Makes simplifying assumptions	☐	☐	☐	☐	☐
M4	Identifies possible variables of interest	☐	☐	☐	☐	☐
M5	Explores relationships	☐	☐	☐	☐	☐
M6	States the mathematical problem	☐	☐	☐	☐	☐
M7	Finds solution	☐	☐	☐	☐	☐
M8	Interprets solution	☐	☐	☐	☐	☐
M9	Validates solution	☐	☐	☐	☐	☐

COMMENTARY

M1	Understands the problem statement and expreses it as objectives to be achieved.
M2	Considers features that are relevant to meeting the task's objectives.
M3	Makes realistic, reasonable and relevant assumptions that are within competence. Justifies and explains assumptions made.
M4	Selects, defines relevant variables, attaches appropriate symbols, units.
M5	Relates the problem to existing knowledge, designs appropriate experiments, if necessary, collects, summarizes and analyses data, constructs empirical and/or theoretical models.
M6	Gives a clear statement of the mathematical problem.
M7	Selects and uses appropriate methods without error.
M8	Explains solution within context of original problem, revises model.
M9	Compares predictions of model with observation and/or common sense.

APPENDIX 2

Statistics

S1 identifies main objectives of the task
S2 Formulation: simplifying assumptions made
S3 Carries out experimentation or surveys
S4 Identifies possible variables of interest
S5 Possible relationships between variables, or conjectures, explored
S6 Makes a mathematical or statistical statement of the problem or conjecture
S7 Finds a solution or gives a proof
S8 Reflects on the investigation

Communication Skills (Written)

W1 Gives a free standing summary or abstract
W2 Gives an introduction to the report
W3 Structures the report logically
W4 Makes the structure of the report verbally explicit
W5 Demonstrates a command of the appropriate written language
W6 Visual presentation and layout complements logical structure
W7 Makes appropriate use of references and appendices
W8 Gives a concluding section in the main report
W9 Gives a well reasoned evaluation

Communication Skills (Oral)

O1 Rapport with the audience
O2 Effective delivery
O3 Command of spoken english
O4 The structure gets over their main points
O5 Clear explanation of the problem and its outcome
O6 Overall planning and organisation
O7 Appropriate use of visual and other aids
O8 Technical quality of visual and other aids

Descriptors O1-O4 are for individuals in a group or for individuals making an individual report.
Descriptors O5-O8 are group descriptors for the whole group or additional descriptors for individuals making an individual report.

11

Assessing Mathematical Comprehension

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1. SUMMARY

This paper examines the rationale for comprehension tests in mathematics, and outlines possible aims and objectives. It describes the author's experiences in setting and using such tests, and, outlines the extent of their use in secondary and higher education in the United Kingdom. References to articles used are given and there is a discussion of student performance in taking such tests. The paper is based to a large extent on articles published by Houston (1993 a,b), and it also contains new research findings and further reflection.

2. RATIONALE

Comprehension Tests in Mathematics are a means of developing and assessing a student's ability to read and understand a published article on mathematics or an application of mathematics. They are a means of encouraging students to develop independent learning skills which will, in turn, bring students to a greater understanding of what they have learnt.

The methodology employed by the author is to give to students a copy of a published article which describes a mathematical modelling activity. The students are asked to read the article carefully, to work through it with pencil and paper, to note the problem statement, simplifying assumptions made and the details of the model created. They are to repeat at least one of the calculations and to reflect on the interpretation of results and validation of the model as reported in the article. They are to question the assumptions made and conclusions drawn. They are encouraged to discuss the article with their peers.

Then after a set time, usually three weeks, the students take an unseen written examination which asks questions about the article. What are the assumptions made? Are they justified? Are there any errors in the calculations? and so on ... If the students are well prepared then the questions will not come as a surprise to them. Nevertheless it is the author's experience that this test is a good discriminator, producing a reasonable spread of marks.

Articles on some aspects of pure mathematics could equally well be used in a different context and questions asked to draw out the student's understanding of the mathematical process involved.

This activity is useful because students will learn, not only the mathematical content of what they have read, but also they will have a greater appreciation of the mathematical processes involved.

If the article is about an application of mathematics, they will learn something of the wider background of the problem being solved thus seeing that mathematics is a powerful problem-solving tool, that it is a living subject, and that it is relevant to everyday life. Furthermore the process of preparing for the test encourages group discussion and cooperation.

There is now widespread international agreement that students should be encouraged to read and write mathematics and to ask and answer questions about mathematics.

In the United Kingdom, the proposed core syllabus for A and AS level mathematics published by the School Examination and Assessment Council (SEAC) contains the recommendation that pupils should be able to "read and comprehend a mathematical argument or an example of the application of mathematics" (SEAC, 1993).

Similar encouragement has been given in the United States of America by the National Council of Teachers of Mathematics (NCTM). In Standard 2 of Curriculum and Evaluation Standards for School Mathematics, the NCTM recommends that pupils should be able to "read written presentations of mathematics with understanding", and "ask clarifying and extending questions relating to mathematics they have read or heard about" (NCTM, 1989). NCTM also recommends that "assignments that require students to read mathematics and respond both orally and in writing to questions based on their reading should be an integral part of the (grades) 9-12 mathematical program".

The same encouragement and opportunity should be given to undergraduate students, and this should start in their first year. It will, after all, only be a continuation of practices developed in school. Professional mathematicians spend a lot of time reading other people's work, trying to make sense of it, making it "their own", asking questions about it and working out answers to these questions, looking for mistakes and thinking about generalisations and other applications. It is important that students are introduced to the way of life of the professional as soon as possible.

Even those students–perhaps especially those students–who are not going to be professional mathematicians should also be encouraged to "read mathematics with understanding". While they may never work as creative mathematicians they will almost certainly use mathematical models created by others. For such people to be fully informed citizens and competent professionals in their own field they need to be aware, at least to some extent, of the mathematical modelling process. They should know that a model has been created and is being used in a particular situation, that assumptions have been made which will determine the usefulness of the model, and they should know something about validation. (Houston and McClean (1993) have suggested a universal 6th form course which would help achieve this). They should be prepared to criticise other people's mathematics, and to make up their own minds about things. They should observe critically what other people have done and how they have done it, thus enhancing their own learning of mathematical processes.

Comprehension Tests in Mathematics are one way of encouraging students to "read with understanding". They can be used to test a student's understanding, not only of mathematics itself, but also of mathematical processes, whether processes of mathematical modelling or pure mathematical investigation. They can be used to help students observe critically what other people have done.

While it could be said of academic mathematicians that they are well practised in "reading mathematics with understanding", the same perhaps, could not be said about school teachers or students training to be school teachers. So these groups of people are another target for comprehension tests through either inservice or pre-service training. It is important that teachers should sample, as far as possible, the learning experiences that they will be giving their pupils. So teachers and student teachers should be given comprehension tests to do. But more than that, they should be given instruction and practise in the setting (or writing) of such tests. There is also a new challenge here for mathematics educators in that they have to assess their students' attempts to assess their pupils!

The author has used comprehension tests with senior high school pupils, first year undergraduate students, and school teachers taking an in-service diploma course in mathematical modelling.

3. AIMS AND OBJECTIVES

Houston (1993 a,b) has suggested that the aims of comprehension tests are

(i) to encourage students to read, with understanding, a mathematical article,

(ii) to provide students with an opportunity to demonstrate their wider understanding of general mathematical processes, both pure and applied,

(iii) to encourage students to develop their skills of communicating mathematics-reading, writing, asking and answering, and

(iv) to demonstrate to students that mathematics is a living subject and is used in contemporary situations.

Some assessment objectives of comprehension tests are that students should be able to:

(i) explain all statements like "it can be shown that..." or "it follows from the above that ..." in the article,

(ii) identify and explain all mathematical modelling assumptions made in the article,

(iii) make constructive criticisms of assumptions made, mathematical analysis and calculations carried out, inferences and deductions made, processes carried out,

(iv) locate any inconsistencies or incorrect deductions made in the article,

(v) locate and correct any mathematical or typographical errors in the article,

(vi) have some wider background knowledge of the situation described in the article,

(vii) generalise the ideas or apply the ideas to a different situation.

4. SETTING COMPREHENSION TESTS

Articles selected for use in comprehension tests should provide scope for most of the above objectives to be tested and the tests themselves should include questions which address most of these objectives. The tests can be set as timed, written examinations. This ensures that the answers written on the day are each student's own. For undergraduate students two hours is a reasonable time and questions should be set to ensure that two hours is ample time. In their preparation to take the test, students are encouraged to set their own test questions based on the aims and objectives and on the questions about the article that come into their mind as they work through it. Usually they have been fairly good at this and most of the test questions are not a surprise. Sometimes some of their questions are not asked and they are disappointed; sometimes they gloss over some statement and miss its importance. Another way of operating is to give students the questions at the same time as the article and to require them to provide written solutions after a certain time, say three weeks. This way has two drawbacks. First, it is possible for students to discuss the article and the questions with their peers (which is to be encouraged), but then they could return shared answers which they have not made their own (which is not to be encouraged). This practise does not encourage deep learning, and, of course, it casts doubt on the validity of the test. Secondly, students will not think so much about the article than they would if the questions remain unseen. They will only do what is necessary to answer the given questions. A comparison of results obtained using these two methods with the same test and different groups of similar students is given below.

When teachers or student teachers are asked to set a comprehension test, then additional criteria need to be introduced. Not only do they need to set questions which test most of the objectives, but the questions must satisfy all the criteria of a good examination, namely, they should be unambiguous, they should be do-able in the time available, they should provide opportunities for all students, from weakest to strongest to demonstrate what they "can do", and they should differentiate between weak and strong candidates. Furthermore they should be asked to prepare specimen solutions as part of the whole assignment and these are also assessed for accuracy and relevance. Indeed these solutions provide valuable insights into the intended purposes of the questions.

The inservice groups of teachers were not asked to select an article. Two suitable articles were supplied and they were asked to select one of these. There were two reasons for this. First, selecting a suitable article is a time consuming business and is probably the hardest part of the exercise. Many are read and rejected before a suitable one emerges, so it is useful to try to identify suitable articles throughout the course of one's reading of the literature. These students were part-timers and did not have the time to undertake such a search. Secondly they did not have the resources in their schools and would have had to spend a lot of time in the university library. These obstacles will not be so difficult to overcome when they come to use comprehension tests regularly in their teaching.

Usually a skim read of an article will indicate whether it will be suitable or not for a test. Having selected an article, it should then be worked through "with pencil, paper and computer". Aspects that lend themselves to questions should be jotted down, particularly statements that the author makes about aspects of the modelling process–assumptions, calculations, validation, revision, etc. Students should be given the opportunity to make constructive criticisms of assumptions, methods, calculations and inferences. They could be questioned about the wider background of the article and they could be asked to apply the ideas to a novel situation. The articles discussed in detail by Houston (1993 a,b) are "Handicapping Weightlifters" from Burghes *et al* (1982) and "Disc Pressing" from Edwards and Hamson (1989). Other useful sources of articles are Giordano and Weir (1985), Hart and Croft (1988) and Huntley and James (1991).

5. USE OF COMPREHENSION TESTS

Houston (1993 a,b) has been using comprehension tests in modelling courses at the University of Ulster since 1986. Others including Berry (private communication) and Goldfinch (1992), have also been using such tests with their students.

Comprehension tests have been used in the Northern Ireland Further Mathematics (Mode 2) Examination from 1987 to 1992. See Fitzpatrick and Houston (1989), Greer and McCartney (1989), Houston (1989, 1992), McCartney (1990) and Holcombe (1982). They are also used in the examinations for the Schools Mathematics Project course, SMP 16-19 Mathematics. See Dolan (1988) and Dolan *et al* (1991). These examinations are for pupils in their last year of secondary education.

These authors report that comprehension tests are achieving their stated aims and objectives. Houston (1989) reports that the circulation of the article to pupils in advance of the examination was "a catalyst for group discussion and interaction". McCartney (1990) quoted a teacher who commented on the "very cooperative spirit" in which his class had prepared for the test. Houston (1993a) writes that the whole exercise "seems to have met the aims ... in that students were encouraged (by the threat of an examination) to read a mathematical article. They demonstrated that they had read it with understanding and that they had a reasonable grasp of the important aspects of the mathematical modelling process".

6. STUDENT PERFORMANCE

Houston (1993a) describes in detail how his students answered the questions in the 1992 test. There were three groups of students who took the "Weightlifters" test in 1992, BSc Year 1, HND Year 2 and Postgraduate Diploma (INSET course). The BSc students are better qualified at entry than the HND students but these students have a year's greater maturity. The PG Dip students are all school teachers. Their performance in the test is summarised in Table 1.

Class	BSc 1	HND 2	PG Dip
No. in Class	17	14	16
Range of Scores	18-88	25-55	35-100
Mean Score	50.6	42.0	68.4
Standard Deviation	19.8	8-9	21-6

Table 1: Percentage Scores on "Weightlifters" test

The BSc students performed better on average than the HND students. This was a little unexpected because the level of mathematics required was not great. The HND class fared badly on questions which required explanations. The class of teachers did best of all. This was expected because they brought a greater degree of "worldly wisdom" to the test than the younger students, and a greater command of written English, which enabled them to give good answers to "explain" and "criticise" questions. It is clear that mathematics students need both good numeracy skills and good literary skills to function satisfactorily in mathematical comprehension tests and in those aspects of their professional and private lives where comprehension is required.

Student performance in the 1993 tests was reported at ICTMA-6 and the scores are detailed in Tables 2 and 3. The article used was "Insulating a House" by John Berry, taken from Huntley and James (1991).

Groups of students similar to those who took the 1992 test, took the 1993 at the University of Ulster (Table 2). Performances similar to 1992 were given by BSc1 and PG Dip students. The HND class of 1993 performed better.

Class	BSc 1	HND 2	PG Dip
No. in Class	26	14	9
Range of Scores	23-85	35-73	43-83
Mean Scores	52.2	53.7	65.0
Standard Deviation	12.6	10.4	11.9

Table 2: Percentage Scores on "Insulating a House" Test (University of Ulster)

However the interesting comparison is between the performance of the 40 undergraduate students at Ulster with 37 BSc 1 students at the University of Plymouth, where Berry (private communication) administered the same test but in a different way. Berry gave his students the article and the test questions at the same time and asked them to present written answers after three weeks. As would be expected, a higher mean score was achieved at Plymouth, but there was a greater spread of marks and (almost) a bi modal distribution with 7 students scoring marks in the 30-39 range and 6 students scoring in each of the ranges 70-79 and 80-89, This suggests that students can perform better if they have an opportunity to discuss the actual questions while writing their answers. It also suggests that some students were incapable or lazy.

No. in Class	40	37
Range of Scores	23-85	5-95
Mean Score	52.7	59.6
Standard Deviation	11.9	22.1

Table 3: Percentage Scores on "Insulating a House" Test (Comparison of Ulster and Plymouth)

To give readers an indication of the questions that can be asked in a comprehension test, the questions relating to the "Insulating a House" article are given in the Appendix. The article itself if too long to reproduce here and readers are referred to Huntley and James (1991) pages 81-96.

Houston (1993b) has also reported on how a group of teachers on an INSET course fared in setting a comprehension test. While they set questions to test the arithmetic problems in the article and questions to extend candidates and take them into new situations, the most glaring omission from their tests were questions relating to the mathematical modelling process. This was disappointing considering these teachers had spent a semester studying models and engaging in mathematical modelling. They had not transferred these new ideas to the other domain of activity, namely setting a test. Perhaps they were too recently introduced to modelling to achieve this, or their instructor had not sufficiently emphasised objectives (ii) and (iii)!

7. CONCLUSION

This article has given a rationale for using comprehension tests in mathematics. It has outlined aims and objectives and described how such tests can be constructed and used. It has discussed student performance in taking instructor devised tests and in devising tests themselves.

It is concluded that these activities are meeting the stated aims, particularly the aim of encouraging students to read, ask, answer and write mathematics, with at least some understanding!

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APPENDIX

These questions relate to the modelling article "Insulating a House" by J. S. Berry, in Huntley and James (1991), pages 81-96, which should be read in conjunction with this appendix.

The questions refer to the different lines of this article by number and these need to be inserted by the reader for ease of reference.

line 1 is the line "7.1 The Problem Statement" on p. 81
line 50 is the last line on p. 82
line 100 is the line "17. thermal properties of walls" on p. 85
line 150 is the line "properties of materials are modelled by a simple expression" on p. 87
line 200 is the line "Area of glass = Ag" on p. 90
line 250 is the line "Fig. 7.6 shows a graph of Pg/Pb against x" on p. 92
line 281 is the last line before "Exercise 1" on p. 95

(i) In line 1 it is claimed that "heating a house or flat is an expensive part of the weekly budget". State whether you agree or disagree with this statement, giving reasons for your answer. Include appropriate facts and figures.

(ii) Referring to line 8, are there any other routes for heat escape?

(iii) Referring to Fig. 7.1, is the greatest heat loss per unit area through the walls, the roof or the windows? Estimate areas for these routes.

(iv) Referring to Fig. 7.2 and your answer to question (i), estimate the cost of cavity wall insulation. Does this concur with your other enquiries/experiences or the date in Table 7.1? Explain your answer.

(v) Describe briefly the difference between "replacement windows" and "secondary double glazing". (Refer to lines 108-110).

(vi) Define "U-value". (Refer to line 157).

(vii) List the assumptions made in creating the heat transfer model. Refer to lines 136 to 189).

(viii) Comment on the statement in line 205.

(ix) In line 221 why does it make sense to call P the "payback period"? Explain how P could be defined differently to take account of the hidden economic costs (e.g. cost of borrowing the capital, or loss of interest that might otherwise have been earned).

(x) Indicate how the value 0.0726 is obtained in line 235.

(xi) Correct the misprints in lines 245, 246.

(xii) Why do you think $h_2 > h_2$?
Why do you think $h_c << h_1$ and h_2?
(Refer to the Diagram for Exercise 2)

(xiii) Do exercise 2.

(xiv) Do exercise 4.

(xv) Which feature of double glazing does most to reduce the U-value? (Refer to question xiv).

12

Assessment and Mathematical Modelling

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SUMMARY

The SIMMS Project is making an effort to motivate and develop mathematical topics through modelling situations presented in context. The author serves as co-chair of the Assessment Committee, so this paper reflects the Project's approach to the assessment of mathematical processes. The Project is described, a scheme for assessment is given, and two examples from the curriculum illustrate the role of assessment in mathematics instruction.

1. THE SIMMS PROJECT

To put our concerns in context, I will start with a brief description of the project. The Systemic Initiative for Montana Mathematics and Science (SIMMS), a cooperative project of the state of Montana and the National Science Foundation (NSF), is funded through the Montana Council of Teachers of Mathematics (MCTM). The project began in October 1991.

The goals of the SIMMS Project include: (1) to redesign the 9-12 mathematics curriculum (ages 14-18) using an integrated interdisciplinary approach for all students, (2) to develop curriculum and assessment materials, and (3) to incorporate the use of technology in all facets and at all levels of mathematics education (The SIMMS Project, 1993a). The SIMMS curriculum is currently being written by secondary teachers for grades 9-12 mathematics. It is based in applied contexts, integrates the disciplines within mathematics, and incorporates technology as a vital tool.

The SIMMS project makes a distinction between the terms *assessment* and *evaluation*. We consider assessment to be the collecting of data at any level: the system with its institutions and rules; the curriculum with its materials and methods; the classroom with its practices; or the students with their achievements and attitudes. If someone chooses to place a *value* on those results, then they have made an evaluation. For the most part, we are interested in the reliable collection of data, so we consider ourselves an *assessment* component. The student assessment materials that are developed for use in the project are guided by three basic purposes:

1. Assessment informs students of the mathematical outcomes they are expected to achieve. It is through assessment that students learn what we think is important. If something is not assessed, students will not bother to learn it.

2. Assessment informs teachers of the instruction that must be provided so that students can reach the desired outcomes. Assessment must be done on a day-to-day basis, so it must be built into the instructional materials.

3. The results of assessment are used to document the progress of students in meeting the outcomes. We are not advocating the end of grading, but we need models to translate assessment data into evaluation marks. A Model for Instruction and Assessment of Mathematical Applications

2. MODEL FOR INSTRUCTION AND ASSESSMENT OF MATHEMATICAL APPLICATIONS

To discuss the processes encountered in mathematical applications, I will adopt a model that has proved useful in discussing the issues we would like to promote by using mathematical modelling as a teaching method. Although this model is not new, variations have been used for over twenty-five years (see e.g., Klamkin, 1968, and Galbraith & Clatworthy, 1990), it has done such a fine job defining the issues that I continue to rely on it. The model attempts to illustrate the relationships among the following processes:

Solve	Solving the problem is the ultimate goal of an activity. By definition, this is what makes the question a problem.
Simplify	It is usually easier to work with a simpler problem, one that a student is more likely to solve. Students must learn to recognise and constructproblems of a similar form.

Mathematize Students must construct a "mathematical model" that is isomorphic to the real world situation.

General The processes of communication, inquiry, and reflection are appropriate for all of the other dimensions.

The first three processes mentioned are reversible, and each reverse direction identifies a desired skill as important as the original:

The opposite of **solve** is to **verify**.
The opposite of **simplify** is to **generalize**.
The opposite of **mathematize** is to **interpret**.

Each of the processes listed (along with its opposite) can be regarded as a dimension in a cube, resulting in the 3-dimensional model pictured in Fig. 1.

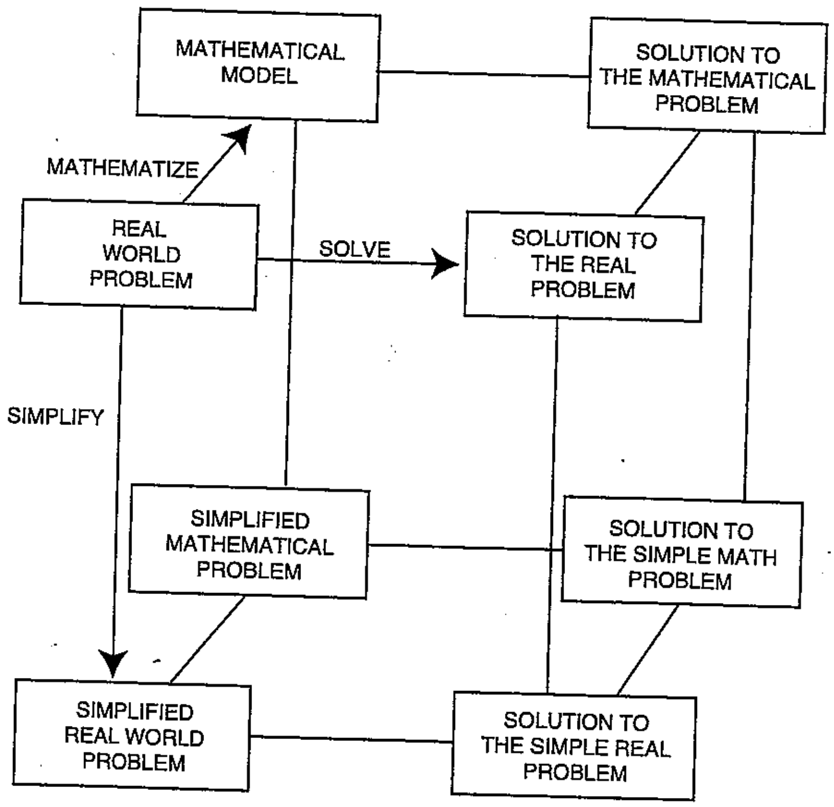


Fig. 1 A 3-dimensional Model

Activities for approaching mathematics in context take place in a variety of dimensions. The most elementary activities occur on the corners, such as (1) to describe and classify, (2) to compare and order, (3) to join and separate, and (4) to group and partition. These elementary activities can be done using "real world" objects like blocks and sets or they can be done using "mathematical world" objects like numbers and relations.

More complex activities are represented by the edges of the cube. One edge, the solving and verifying dimension, often employs the traditional tools of mathematics: finding equivalent statements, applying algorithms, validating empirically, and employing logical (deductive) arguments. These processes, also, can be applied to either "real" objects or "mathematical" objects.

The simplify and generalize dimension may involve changing modes of representation, suppressing detail, and extrapolating or expanding the domain to new situations. For example, we often ignore statistical error and model a set of data using a straight line. It is important to ask what is gained and what is lost by making such an assumption.

The mathematize and interpret dimension involves the construction of mathematical models that reflect all of the essential properties of the real situation. Several examples would illustrate the issues raised along this dimension, but I will use just one. How do we define, measure, and discuss the expectation of waiting time at a traffic light? What variables are relevant? What variables can be ignored? One possibility is to graph the length of waiting time against the time of arrival. Fig. 2 shows a graph of the time spent waiting upon arrival at a light that is red for 60 seconds then green for 30 seconds. Even given this mathematization of the situation, we still must address the relationships between the "real" and "mathematical" models. What is the question? What are we looking for? How can we represent the "real solution" within the "mathematical model?"

Another critical point in mathematical modelling is the importance of general processes: communication, inquiry, and reflection. These are processes we want to encourage with mathematical modelling activities, but they are related to all of the dimensions. Communication is required if students are to understand and explain their approaches. A valuable technique that can be applied to both "real" and "mathematical" situations is to explore and inquire about the phenomena under investigation. Reflection is the process by which we look back and refine the procedures that we have developed.

In previous discussions of this model (Hirstein, 1991), I have argued that activity also takes place on the planes of this cube. I will give only two examples because I think most of these issues have been discussed.

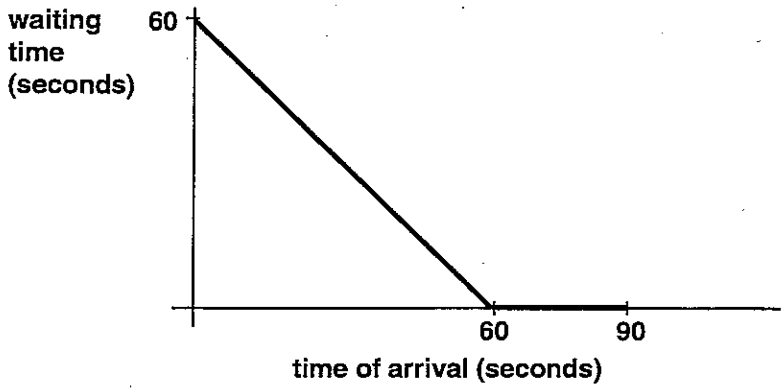


Fig. 2 A Graph of Waiting Time

The first example concerns those people who are content to live in the "back plane" of the cube, that is, in the "mathematical world" containing only the simplify and solve dimensions. When I was a young teacher, the only mathematics teachers I knew lived in that plane. Real world uses of mathematics were not considered necessary for the development of the mathematical systems and techniques that were taught. Today, largely through the efforts of groups like those represented at this conference, the percentage and the influence of those who would ignore applications in the teaching of mathematics are decreasing.

The second example concerns the issue of the "top" plane versus the "bottom" plane. I used to argue that these planes are isomorphic, so "Why be concerned with the difference?" I still think they are isomorphic, but I now recognise that "simplicity" is a function of the person perceiving the problem and that when novices are presented with significant problems (and they should be), it is almost always a good idea to play with a "simpler" version of the problem to get a feel for the situation. For instance, studying lotteries generates some very complicated problems because they deal with large numbers. However, after exploring the situation with smaller numbers, students can develop strategies for explaining the mathematics of chance, expected value, and the probability of winning. These concepts can then be applied to analyse the more complicated "real world" problem.

3. AN ASSESSMENT SCHEME

If the processes of mathematics described in the 3-dimensional model are to be valued by students, instruction and assessment will have to reflect the new process outcomes. Alternative forms of assessment can be developed to create a multi-dimensional profile of a student's development of the use of mathematical processes over time. New instruments that include open-ended items, extended problem solving situations, student demonstrations, real-world based explorations need to be developed at all levels of school mathematics. Assessment depends on a student's communication of the modelling processes, on a student's approach to the problem, and on the ability to reflect on the solution. Awareness of the modelling dimensions and their assessment criteria will help students determine what is important and make student self-assessment possible. The purposes of the following criteria include identifying and improving student responses on these dimensions.

The Physical Situation	The student understands the problem context and its structure, and recognises subproblems.
Mathematizing	The student uses an identifiable and appropriate model on the problem (may use tools such as diagrams, graphs, variables).
Using the Mathematics	The student makes progress toward a solution using the model.
Interpreting	The student interprets solution(s) and validates the mathematical model.

The chart pictured in Fig. 3 is suggested to record the marks for individual responses assessed using the four criteria above. Scoring guidelines, providing a set of examples that show a range of responses together with instructional implications, need to be developed. The sample responses are also useful in defining the issues that lead to a consistent framework to assess student papers.

4. TWO EXAMPLES FROM THE SIMMS CURRICULUM

I will use two examples from the SIMMS Project Level 1 materials to illustrate the nature of the activities and to discuss the issues that are raised by our assessment objectives. The first example, shown in Fig. 4, is a Summary Assessment question from a module on the geometry of reflections.

This example illustrates one change in the classroom, the use of assessment in instruction. The students here are inventing the questions. In one classroom, the teacher used this activity to motivate the students to solve the problems they had written. One golf hole from each student was photocopied and given to each team of four students to solve the next day. Although several of the problems were relatively difficult, all of the students were challenged to solve the problems that had been posed by their peers. Some of the golf holes designed required innovative solutions, and often led to detailed analyses and discussions.

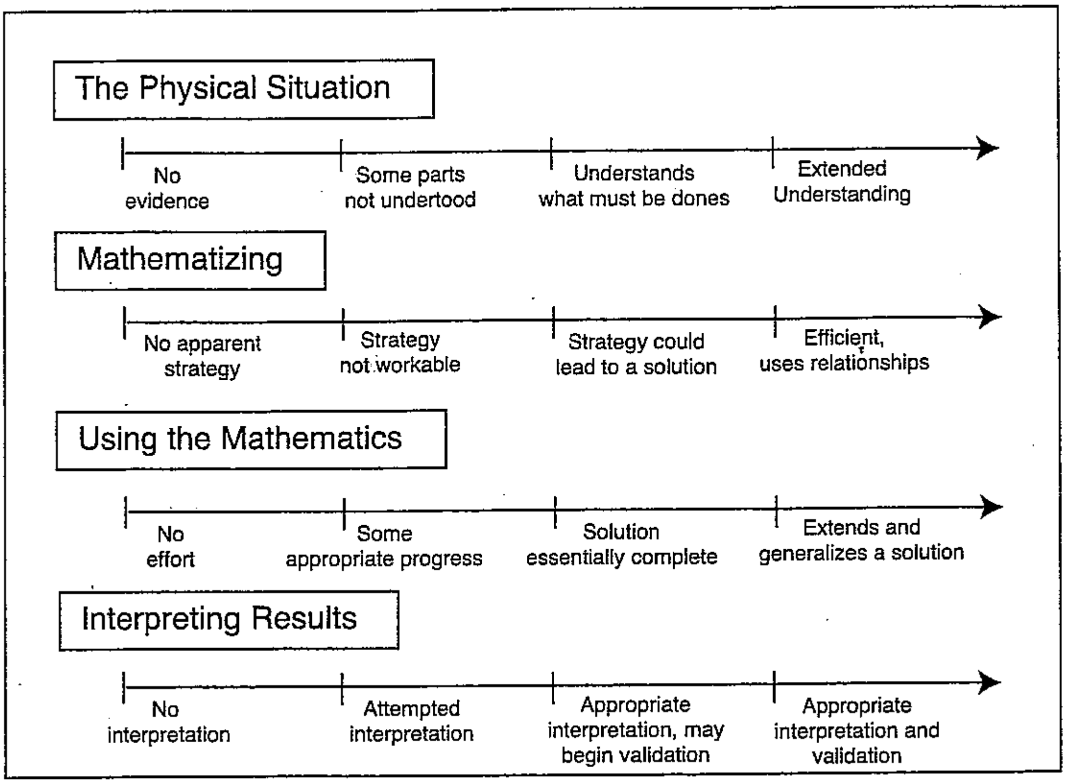
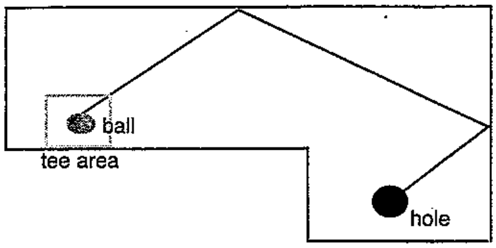


Fig. 3 An Assessment Chart

Activities like this one provide a wonderful existence proof about how applications can enhance classroom discourse. However, this example also illustrates another important issue about applications: model selection. One teacher made this activity into a laboratory experience in the corridor. Students used a wooden board to make a boundary, then tried to bounce a golf ball off the board to hit a target. The students were able to solve the problem using a strategy similar to reflecting light in a mirror: find the reflected image, then draw the straight line of light to find the point of intersection on the boundary. In a later conversation, the teacher was asked if the same problem could have been done using the baseboard of the wall in the corridor. This extension had been considered and rejected because the students would not be able to find the reflected image on the other side of the wall. However, this additional restriction results in a beautiful extension of the original problem, even though the solution requires a different mathematical analysis of the question.

Most miniature golf courses have holes requiring a person to hit the ball off at least one wall to score a hole-in-one. And example is shown below:



Design and draw some miniature golf holes. There are three rules.

1. The drawing must be to scale, with all dimensions indicated.
2. Only line segments may be used for walls.
3. A tee area must be provided.

As you design holes, sketch possible paths for a ball. For the final drawing of a hole, however, do not reveal your winning strategy. Design the following types of holes:

a. A hole that looks simple, but where a hole-in-one is probably impossible.
b. A hole that looks difficult, but has a simple path for the ball.
c. A hole that has many poaaible paths.
d. A hole that requires a golfer to bank the ball off exactly three walls to get a hole-in-one.

To test your designs, trade drawings with a classmate and try to sketch the paths of a hole-in-one. After the designs have been tested, present one of them to the rest of the class. Use mathematical ideas and the language of this module to explain your design.

The SIMMS Project, 1993b

Fig. 4 A Reflection Example

The modelling process should be the critical object of instruction here. The students' desire to model the new situation should be so prevalent that they do it automatically. Students must not feel impeded just because a first model doesn't work. As teachers, we should welcome questions that force us (and our students) to seek alternatives and gain confidence in our ability to mathematize one situation with several models. The importance of these activities can only be communicated to students by providing feedback on all four dimensions of the assessment chart (see section 3).

How far will 100 barrels of oil spread? To investigate this question, you may simulate an oil spill with water. You will need several containers and enough water to fill the container with the least volume.

Each container of water models a spill at some time while the oil is spreading. The same amount of liquid is placed in each container. You will investigate the relationship between the area of the base of a container and the height of the liquid.

a. Prepare a table with three headings: area of base, height and area • height.
b. Identify the container that holds the least volume. Determine and record the area of its base and its height. Record both. Calculate the volume of this container (base area • height) and record it.
c. Fill the container having the least volume with water.
d. For each of the unfilled containers, determine the area of the base, record it in the table and arrange the remaining containers from least base area to greatest.
e. Carefully pour all of the water from the container with the least volume into the next larger container. Measure and record the height of the water next to the base area for that container. Repeat for each container.
f. Graph area of base versus height.
g. Save the results of this exploration for use in the assignment.

The SIMMS Project, 1993c

Fig. 5 An Inverse Variation Example

The second example, shown in Fig. 5, is an exploration that illustrates inverse variation. It is taken from a module on oil spills. An oil spill is a volume of oil that is spreading over a large area. The relationships among the volume, the area, and the thickness of the oil are critical to predicting the behavior of the spill. This activity uses a fixed quantity of water in a variety of containers to investigate the relationships in the situation, then later compares the model developed to the behavior of oil.

This investigation results in data that are modelled by inverse variation. The same quantity of water should be the product of the base area and the height of the water in each of the containers. Measurement error can lead to a wide discrepancy in the data. The solution of one group of students is given in Fig. 4. These data were obtained from eight cylindrical containers, then the students graphed the height of water against the base area of the container.

In the chart of Fig. 6, base area and height are direct measurements; volume is the product of the other two columns. Errors in measurement resulted in a wide variance in the "constant volume", and even one case where the function has two y-values for the same x-value. These difficulties are common when data are collected by students, but they are not often found in textbook data. Mathematics students are not sure how to approach these problems. In fact, when one student group made an error placing one point far off of the curve, they concluded that the inverse variation model was inappropriate. Science students are encouraged to question the data, perhaps to check their measurements. Mathematics students tend to accept the data without question (once

shape	basearea	height	volume
cylinder	9.62	8	76.96
cylinder	12.56	5.5	69.08
cylinder	19.625	3	58.875
cylinder	38.465	2.5	96.162
cylinder	15.89	3	47.67
cylinder	28.26	2	56.52
cylinder	19.625	3.5	68.6875
cylinder	23.75	2.5	59.375

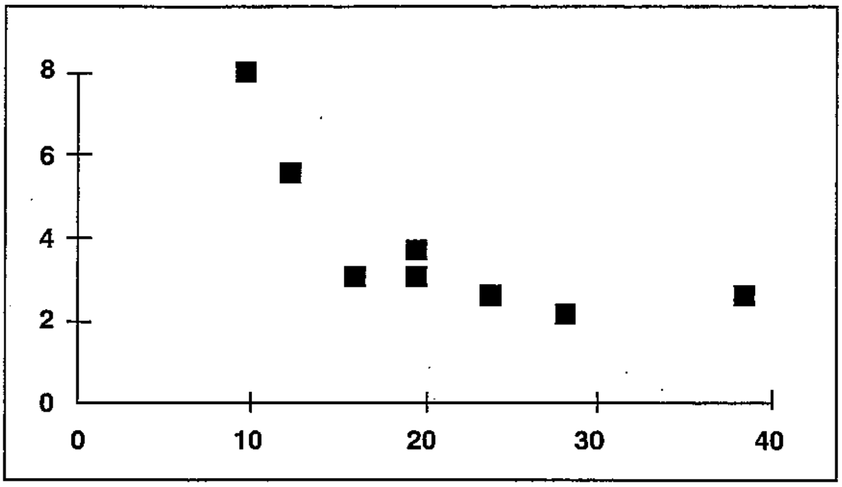


Fig. 6 Student Response to the Oil Activity

collected), so the model must be wrong. The approach to a modelling activity must include a validation of the model. Here again, these goals, are communicated by scoring the class responses on all of the dimensions of the assessment scheme.

5. CONCLUDING REMARKS

One of the most difficult aspects of assessment is getting it included in instruction. Traditional mathematics materials have made it very difficult to assess students' higher-order thinking on a daily basis. There are basically two difficulties:

1. Assessment usually comes at the end. You give a test, write the number down, and go on to the next chapter. This, of course, is an exaggeration, but most assessment occurs after it's too late to do anything about it. We are trying to change this attitude by placing assessment opportunities directly within the curricular

materials.

2. All students must be treated alike, so all students must have the same assessment opportunities **on the same items**. Therefore, new multi-dimensional assessment becomes over burdensome. Teachers are rightfully concerned about doing "that much" with every student, every day. We have to accept scoring schemes that allow a teacher to look at a few students' work each day and rotate questions among students. All students do not have to respond to the same questions, as long as all students have an opportunity to exhibit the same achievement.

Our assessment writers have suggested a scheme that addresses four dimensions of the mathematical modelling process. We tried to keep the number of dimensions small, but not one. If we want students to achieve something, we know we have to assess it. Only through our assessment do students come to know we think something is important. But it is equally critical that we use assessment to find out what students can and cannot do. The results of assessment must feed back into our instructional decisions. Now that we are collecting examples of student work **from our project**, we are beginning to ask teachers to help determine the criteria that represent a shared meaning of assessment. This alone gives us an optimistic feeling that change can happen.

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13

The Assessment of Core Skills in the Context of Mathematical Modelling

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SUMMARY

Current initiatives in the UK, within both education and employment sectors, have aimed at improving skills of young people through the definition of occupational competence. National occupational standards have been developed that seek to ensure that employees and potential employees possess the necessary background knowledge in their chosen area of work, but can also demonstrate a range of core or common skills.

The BTEC (Business and Technology Education Council) Higher National Diploma in Computing Mathematics at Sheffield Hallam University, is one course in which an assessment strategy has been developed that ensures that students demonstrate both mathematical and core skills.

This paper describes, through the use of a case study, how a new assessment strategy has been introduced into the modelling unit undertaken by students towards the end of their first year on the course.

1. BACKGROUND

Common Skills in the National Context

Traditionally, qualifications in higher education have been knowledge-led. Courses have been built around subject areas, sometimes divided into units or modules. The current growth of combined studies courses using the "pick and mix" philosophy of Credit Accumulation and Transfer (CATs) is reinforcing the idea of a course being a collection of possibly disparate units of different, specified levels.

Traditional assessment methodologies have been built around time constrained unseen examinations, which examine only a proportion of the defined syllabus, and require only a percentage of "correct" answers in order for the student to pass the exam. Increasingly, however, there is a move towards an assessment process that enables examiners to assert that a person who has successfully completed a unit has been assessed and found competent in all the specified outcomes of that unit. This movement towards outcomes-led assessment has its roots in three major areas:

a) the demand by employers for a more flexible and relevantly qualified workforce

b) the establishment by the Government of the National Council for Vocational Qualifications

c) the recognition by examining and awarding bodies-such as BTEC-that qualifications need to incorporate more than subject specific skills and knowledge alone if the current needs of employers are to be met

Employer organisations such as the Confederation of British Industry (CBI) and Chambers of Commerce have often called for employees to be better qualified in

communication skills
numeracy skills
keyboard skills

in addition to the occupationally specific skills around which qualifications have always been built.

The National Council for Vocational Qualifications, since its establishment in 1986, has been developing National Vocational Qualifications (NVQs) founded on defined national standards of competence in specified occupational roles. More recently, the Council has also developed General National Vocational Qualifications, which describe the skills and knowledge which underpin a range of roles within a given occupational sector. In both developments, the need for skills beyond those that can be described in the performance of tasks inherent in the roles themselves, has been recognised. In the context of GNVQs, distinct units of "core skills" are described and must be assessed in order for the qualification to be awarded (Oates 1992). These are:

self-management
group and team work
communicating
problem solving
numeracy
information technology

In the schools sector, the Schools Examination and Assessment Council (SEAC) and the National Curriculum Council (NCC) who advise the Government on curriculum content have proposed:

"from 1994, A levels should include 'common learning outcomes' in problem solving, personal skills, communication skills, number skills, IT skills and foreign language skills."

In terms of the HND in Computing Mathematics, and in particular, the Mathematical Modelling unit, the greatest influence has been that of BTEC. This awarding body insists that every student, on each of their validated courses, must be assessed in what they term "common skills". These common skills are afforded such importance that failure in any one of them would mean the withholding of the award.

The BTEC common skills are grouped into seven generic areas:

managing and developing self &
working with and relating to others
communicating
managing tasks and solving problems
applying numeracy
applying technology
applying design and creativity

These are further subdivided into 18 more specific competences. For example, the ninth competence, which lies in the area of "communicating" is

present information in a variety of visual forms

Each common skill generic area is defined by

general skill area aims

and each element within a generic skill area is defined by

performance criteria and

range statements

Full definitions can be found in "Common Skills and Core Themes" (BTEC 1992).

Mathematical Modelling

One key to the assessment of mathematical modelling is to recognise that the BTEC common skills identified above are implicitly present in all the stages of the modelling process, one version of which is portrayed in figure 1 (Edwards and Hamson 1989). Modelling is often simplistically viewed as this sequential loop but is more usually an iterated process with many intermediate returns from one stage to an earlier stage and sometimes back to the customer.

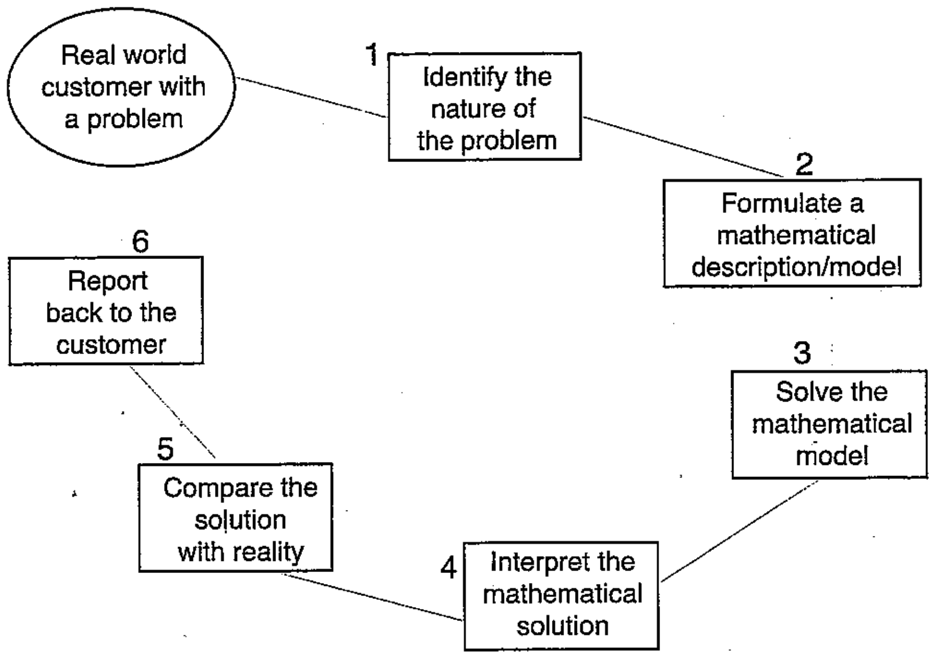

Fig. 1

It may be argued that the most important stages are step 1 and step 6. These are the processes that are critically dependent on communication skills. Within the rest of the loop, we can also identify the need for

a) research skills (stage 1)

b) modelling skills – identifying the mathematical problem (stages 1 and 2)

c) mathematical skills – solving the mathematical problems (stages 3 and 4)

d) IT skills (stages 3 and 4)

e) evaluative skills (stage 5)

f) investigative skills – asking "what if" questions (stages 2,3,4,5)

Modelling therefore incorporates the need for communications at the "front end" of the process of identifying the problem; the mathematics/IT area of making assumptions, building models and reaching solutions; and, finally, communications again in reporting back.

2. AIMS OF A MATHEMATICAL MODELLING UNIT

Modelling is a holistic activity. Its aims include

a) motivating interest in quantitative methods

b) developing group and/or team work

c) integrating and developing IT dependent skills

d) researching information sources using appropriate IT

e) collecting data

f) identifying assumptions and building models

g) applying mathematics (or even numeracy!)

h) communicating verbally and in writing

i) evaluation and critical analysis of solutions

j) improvement of earlier models

k) suggestions for future work

l) applying solutions predictively

This extensive agenda of skills and aims is comprehensively addressed in the following case study.

3. A MODELLING CASE STUDY OUTLINE

Population With Age Structure

Doucet and Sloep (1992) in their chapter on population modelling include a simple example of modelling populations with age structure. A population may be classified into age groups for each of which the birth and survival rates are known with respect to a given time period. If these rates are assumed constant, the numbers of individuals in each age group at each time step $i+1$ in the future can be predicted from the group sizes at time step i by

$$N_{i+1} = AN_i \tag{1}$$

where

> N is the vector whose components are the numbers of individuals in each age class
>
> i is a time step counter
>
> A is the Leslie matrix or projection matrix for that particular population.

The matrix A contains the birth rates for each age class in the first row and the death rates for each age class in the lower sub-diagonal.

A model such as this is the simplest model that could be used to predict the numbers in human population age groups. This would be of value, say, if a government were interested in planning future requirements (e.g. teachers, doctors, state pensions ...).

Given in Appendix 1 is the case study as given to students on the mathematical modelling course. In summary the task is:

> **produce a model of the England and Wales population growth taking into account the birth and death rates that vary according to age**

and use it to develop models that could be used to help in planning future requirements. Note also that the problem specification asks each student to submit a **BTEC Log Sheet** (Appendix 2) which is the reporting mechanism for common skills, along with the more usual group report.

Delivery Mechanism

The case study is deliberately open ended. However, some direction is given to students. At this stage in their course, the students already have a range of experience and knowledge that includes:

a) all the necessary mathematical skills and knowledge

b) library skills

c) report writing skills

d) earlier modelling activities involving group work

e) spreadsheet skills

The modelling exercise is intended to be completed in a period of four weeks, with four hours a week specifically timetabled towards the activity.

In the first session, the mathematical background is presented. Groups of 2 or 3 students then set off to search out relevant data and return when necessary for additional support. The building of the simple internal models at each stage was relatively simple, but there was a need to give help and reassurance to build the students' self-confidence.

The solution of the basic model is the repeated matrix calculation using equation (1). This is easily accomplished using any spreadsheet. The result is a series of tables and graphs of the sizes of the populations in each age group in the future.

Subsequent questions are answered by building additional models depending for their input data on the results already calculated.

4. ASSESSMENT

The assessment of the activity is based on both the report and the completed log sheet and necessitates the disaggregation of the mathematical content from the processes used in carrying out the exercise.

The report presented by the students should demonstrate that appropriate models have been built, that correct numbers have been calculated and suitably presented, and that appropriate conclusions have been drawn. This element of mathematical activity is given scores or grades in accordance with the marking scheme of the unit.

When the case study was introduced, the students were also informed that they would have to:

a) find appropriate data sources

b) think about their problem

c) formulate ways around the problems as they arose

d) think about how to report their numerical findings

e) find interest and application in their mathematics

f) work effectively in groups

g) use skillfully the IT tools available

h) write reports

i) make verbal presentations

The report should reflect their efforts in these areas. It is from this written evidence that the claims made on their log sheets to the specified common skill competences is assessed by the tutor, possibly with formative feedback entered into the tutor comments section.

5. ISSUES OF WIDER CONCERN

Throughout their course, it is emphasised to students that these skills are not only appropriate in the mathematical modelling unit of their course, but are integral to their studies in higher education. Although the above skills were specified to these students in the context of this exercise, students are permitted to claim particular competences at any time during their course, using any appropriate evidence. They do this by submitting BTEC log sheets.

> This student–led approach to assessment has implications for all tutors involved in teaching and assessing BTEC programmes. In order to ensure that the approach to assessment is adopted across whole courses, it is necessary to: ensure that subject tutors are explicit about what they think is involved in each assessed task and to identify which aspects will be assessed
>
> **ensure that each student submits, for each piece of assessed work across all units, a completed Common Skills Log Sheet (Appendix 2)**

When marking the piece of work, tutors will have the evidence (the work presented by the student) in support of the student's claim to competence as described on the log sheet. The submitted work may clearly provide sufficient evidence for the claim to be justified, in which case the tutor simply records this on the log sheet. If it does not, then the tutor records appropriate comments on the log sheet, thus enabling the use of the log sheets for both formative and summative assessment purposes.

Fundamental to the process is the principle that it is the responsibility of the student to present both the evidence and the log sheets, and to retain the latter throughout their whole course of study. In this way, a continuous profile of demonstrated competence in each of the common skills elements is generated by each student. The cumulative results of the common skills log sheets are entered onto the BTEC Common Skills Progress Profile (Appendix 3). This final document summarises development in all the skill areas, and is the basis on which the final common skills grades are awarded. Further details of this process can be found in Common Skills and Core Themes, Implementation Guidance (BTEC 1992).

6. CONCLUSIONS

In accordance with the perceived needs of present day society, common skills are becoming an explicit underpinning theme of higher education in the UK. Employers are distinguishing between potential candidates for jobs on the basis of more than occupationally specific knowledge provided by traditional named qualifications. The development of competence in the so-called common skills is therefore becoming of increasing importance, particularly in first degree courses. The need to undertake this work is also becoming increasingly important with the rapid growth in participation in higher education, and the further expansion of combined studies qualifications.

The explicit development and assessment of common skills can be integrated throughout all units of a course of study. This becomes practical and feasible by using a mechanism like the log sheets and profile forms. Such an approach gives the added advantage of providing a focus for formative assessment through appropriate feedback, whilst increasing the emphasis on student centred learning.

However, implementing this system has implications across the range of delivery mechanisms of any given programmes. While students are increasingly entering higher education with some experience of self-directed learning, there is still some reluctance to engage in activities that are not obviously, in their minds, connected with the subject content of their course of study. Such reluctance to engage in the process of developing common skills may be compounded by staff who are themselves unsure of the demands that will be placed on them. Many of the tutorial staff in higher education have been taught, and have themselves taught, in a system that has not previously laid emphasis on assessment to specified criteria. This, therefore, represents one challenge to them. However, when these criteria describe areas that have not previously been addressed by these tutors, but which now need to be integrated into the subject matter with which they are familiar, there may be practical and emotional barriers to be overcome.

In the mathematical modelling course at Sheffield Hallam University, a teaching and assessment model has been developed that has risen to the challenges and has, in great part overcome them. The lessons that have been learned are:

a) that it is possible to integrate common skills into a case study based on mathematical modelling

b) that students will respond to the need to identify and develop common skills, and will take responsibility for making claims to competence in these areas

c) that busy staff need persuading of the value of their involvement in the teaching and assessment of common skills

d) that the value of undertaking teaching and assessment of common skills becomes apparent to both students and staff when these skills are used successfully in the job market.

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APPENDIX 1

The overall task is to produce a model of the growth of the population of England and Wales taking into account the birth and death rates that vary according to age. This model is then to be run to predict the numbers of individuals in each of the separate age classes for some yearws into the future. The aim is to be able to plan future service requirements as identified below. Your aim is to complete the following tasks.

1. Use an appropriate information source to estimate the **numbers, birthrates, survival rates** of people in the age classes 0-7, 8-15, 16-23 etc. for the population of England and Wales for any reference year from the 1980s. This will itself involve elements of modelling and will need to be explained in your report.

2. Construct the "population iwth age structure discrete time step mode" taking the somewhat crude age classes given above ablong with a basic time step unit of 8 years.

3. Use a spreadsheet (or otherwise calculation to find the numbers for each age class over the next 200 years.

4. Develop models for the numbers:

   i. in full-time education

   ii. of teachers required for (i)

   iii. in post-16 education

   iv. available for work

   v. of people over retirement age

   and apply each of them over the next 64 years.

   In each case state clearly the assumptions you make giving justifications for them and additional references if appropriate.

5. Present the results obtained in 3 and 4 above in graphical form and give general descriptions of their pertinent features. Where possible, compare your results with any other predictions you can find.

6. Write up your activity as a group report. It is not necessary to word-process everything though this can aid legibility. As well as any discussion of results obtained, include a conclusions section

assumptions, accuracy, usefulness, validity, improvements to the mdoels, etc.) and also, briefly, the group working aspects of the activity.

Finally, indicate what you consider a fair breakdown of the marks should be (e.g. a group of 3 putting in unequal efforts could request that the available mark [60% say] was split in the ratio 1:2:2 [36%, 72%, 72%]).

7. Submit individual common skills log-sheets. The claims of competence should be supported by aspects of the written report.

MARK SCHEME

The report as a whole	10%
Part 1	20%
Parts 2 and 3	5%
Part 3	5%
Part 4 i) to v)	5% each
Part 5	20%
Conclusions	15%

Doucet, P. and Sloep, P. B. (1992) *Mathematical Modelling in the Life Sciences.* London: Ellis Horwood.

Central Statistical Office, Annual Abstract of Statistics.

Population Trends

Demographic Year Book

APPENDIX 2

NAME: ______________________ DATE: __________

OUTCOMES REVIEWED AT THIS STAGE	SUPPORTING EVIDENCE	TUTOR'S COMMENTS
MANAGING & DEVELOPING SELF 1. Manage own roles and responsibilities. 2. Manage own time in achieving objectives. 3. Undertake personal and career development. 4. Transfer skills gained to new and changing situations and contexts.		
WORKING AND RELATING TO OTHERS 5. Treat others' values, beliefs and opinions with respect. 6. Relate to and interact effectively with individuals and groups. 7. Work effectively as a member of a team.		
COMMUNICATING 8. Receive and respond to a variety of information. 9. Present information in a variety of visual forms. 10. Communicate in writing. 11. Participate in oral and non-verbal communication.		

NAME: ____________________ DATE: __________

OUTCOMES REVIEWED AT THIS STAGE	SUPPORTING EVIDENCE	TUTOR'S COMMENTS
MANAGING TASKS & SOLVING PROBLEMS 12. Use information sources. 13. Deal with a combination of routine and non-routine tasks. 14. Identify and solve routine and non-routine problems.		
APPLYING NUMERACY 15. Apply numerical skills and techniques.		
APPLYING TECHNOLOGY 16. Use a range of technological equipment and systems.		
APPLYING DESIGN AND CREATIVITY 17. Apply a range of skills and techniques to develop a variety of ideas in the creation of new/modified products, services or situations. 18. Use a range of thought processes.		

APPENDIX 3

BTEC Common Skills: Progress Profile

Review Date								
Tutor's initials								

Centre	
Learner	
Programme	

Skill area and outcomes	*Log sheet/Review reference*								*Final grade to be reported*	*Signatures and dates*
MANAGING AND DEVELOPING SELF										
1 Manage own roles and responsibilities										
2 Manage own time in achieving objectives										
3 Undertake personal and career development										
4 Transfer skills gained to new and changing situations and contexts										
WORKING WITH AND RELATING TO OTHERS										
5 Treat others' values, beliefs and opinions with respect										
6 Relate to and interact effectively with individuals and groups										
7 Work effectively as a member of a team										
COMMUNICATING										
8 Receive and respond to a variety of information										
9 Present information in a variety of visual forms										
10 Communicate in writing										
11 Participate in oral and non-verbal communication										

and groups
Yes 7 Work effectively as a member of a team

COMMUNICATING

8 Receive & respond to a variety of information
Yes 9 Present information in a variety of visual forms
Yes 10 Communicate in writing

MANAGING TASKS & SOLVING PROBLEMS

Yes 12 Use information sources
Yes 13 Deal with a combination of routine & non-routine tasks
Yes 14 Identify and solve routine non-routine problems

APPLYING NUMERACY

15 Apply numerical skills & techniques

APPLYING TECHNOLOGY

16 Use a range of technological equipment and systems

APPLYING DESIGN & CREATIVITY

17 Apply a range of skills & techniques to develop a variety of ideas in the creation of new/modified products, services or situations
Yes 18 Use a range of thought processes

POPULATION MODELLING WITH

AGE DISTRIBUTION

The overall task is to produce a model of the growth of the population of England and Wales taking into account the birth and death rates that vary according to age. This model is then to be run to predict the numbers of individuals in each of the separate age classes for some years into the future. The aim is to be able to plan future service requirements as identified below. Your aim is to complete the following tasks.

1. Use an appropriate information source to estimate the numbers, birth rates, survival rates of people in the age classes 0-7, 8-15, 16-23, etc. for the population of England and Wales for any reference year from the 1980's. This will itself involve elements of modelling and will need to be explained in your report.

2. Construct the "population with age structure discrete time step model" taking the somewhat crude age classes given above along with a basic time step unit of 8 years.

3. Use a spreadsheet (or otherwise) calculation to find the numbers for each age class over the next 200 years.

4. Develop models for the numbers:

   i) in full-time education

   ii) of teachers required for i)

   iii) in post-16 education

   iv) available for work

   v) of people over retirement age

   and apply each of them over the next 64 years. In each case state clearly the assumptions you make giving justifications for them and additional references if appropriate.

5. Present the results obtained in 3 and 4 above in graphical form and give general descriptions of their pertinent features. Where possible, compare your results with any other predictions you can find.

6. Write up your activity as a group report. It is not necessary to word-process everything though this can aid legibility. As well as any discussion of results obtained, include a conclusions section which is to reflect both on the technical aspects (eg. reviewing assumptions, accuracy, usefulness, validity, improvements to the models, etc.) and also, briefly, the group working aspects of the activity. Finally, indicate what you consider a fair breakdown of the marks should be (eg. a group of 3 putting in unequal efforts could request that the available mark [60 % say] was split in the ratio 1:2:2 [36%,72%,72%]).

7. Submit individual common skills log-sheets. The competence claims should be supported by aspects of the your written report.

MARK SCHEME

The report as a whole 10%

Part 1	20%
Parts 2 and 3	5% each
Part 4 i) to v)	5% each
Part 5	20%
Conclusions	15%

REFERENCES

P. Doucet and P. B. Sloep (1992) *Mathematical Modelling in the Life Sciences.* London: Ellis Horwood.

Central Statistical Office, Annual Abstract of Statistics.

Population Trends

Demographic Year Book

Section D
Industrial Collaboration

14

Authentic Applications of Mathematics

Lars Ebbensgaard
Lemvig Gymnasium, Denmark

SUMMARY

It is well-known that there is a common need for textbooks on mathematical models and modelling. The Association of Upper Secondary School Mathematics Teachers in Denmark therefore contacted the trades and industries to investigate if any projects involving math models and modelling applicable at the upper secondary school level were available. Very positive answers were received and a working group was established to adapt and publish the projects. Until now two textbooks have come out of this meeting between industry and school. The process and the result of this work will be the subject of this contribution.

1. INTRODUCTION

The math-curriculum in the Danish upper secondary non-vocational education of course contains the common main topics of mathematics. In 1988, however, it was added that the pupils should work with three so called aspects:

Aspects (both B- and A-level)

1. The historical aspect

2. The aspect of model and modelling
3. The internal structure of mathematics.
   (The Danish Ministery of Education)

This new curriculum was thoroughly referenced at the ICTMA-4 Conference, *Hermann and Hirsberg* (1991). The engagements in these aspects have been very vivid and have caused much activity.

Literature on the aspects fits only with difficulty into the current textbooks. But we saw great pleasure of experimenting by the pupils as well as by the teachers, who jointly should find and choose the content of these aspects.

As far as the model aspect is concerned, it came to mind to study how mathematics is actually used in the trades and industries. It became common that the subject of mathematics was involved in excursions to factories, treatment plants, agricultural farms etc. in order to study how mathematical thinking was used at the different places.

2. COLLABORATION WITH THE INDUSTRIES ON THE PRODUCTION OF TEXTBOOKS.

These initiatives opened the way for very interesting mathematics and many mathematical models. And it turned out to have a great secondary profit, namely that the well-known question, "What is the use of all this?" was given a serious answer. So the opening to the surrounding society, as a supplement to the classical math-education, was extremely fruitful.

The Association of Upper Secondary Schoolmath Teachers in Denmark now applied to the trades and industries to investigate if any projects were available, where mathematical models and modelling are used in a way that they could be used in the teaching of math in the upper secondary school. Very positive answers were received and a working group was established to adapt and publish the projects.

The adoption turned out to be a difficult task indeed, but because of good will from the companies, it also turned out to be exciting and inspiring.

The math-teachers naturally were extremely happy to see "their" mathematics used so directly. No less was the pleasure of the experts in the companies in telling about their work. In their everyday life they lacked the dialogue with other people on mathematical issues. They were not used to having such interested discussion partners in the mathematical field.

The managements in the companies offered time and resources to the engineers, so there was time enough to discuss and work out the ideas. As a result, four of the projects were issued as a textbook with the title *Authentic Applications of Mathematics, Touborg* (ed) 1992.

One part of the book deals with the chemical industries. Here the concept of distillation is important, and we found that this matter should have a publication of its own, *Distillation, Ebbensgaard et al.* (1993).

The contents on chemical matters were made in collaboration between a high school (Lemvig Gymnasium) and the company of Cheminova Agro.

This collaboration between the educational sector and the industries has been very fruitful. The high school has to a higher degree been absorbed in the surrounding society, which again has had a positive impact on the attitude of the users of the school. The parents, the pupils as well as those on the board have found this initiative very relevant and have welcomed it. So it can be very much recommended to all schools and math-teacher groups to make such a contact with the industries.

The publications mentioned have been the basic material for an in-service course with 35 math-teachers arranged by the Math-association. In a further perspective, I think that the collaboration between high schools and the trades and industries could promote in-side education and job-rotation in the entire field.

One of the engineers in this collaboration expressed the following in the yearbook of the school, "The postulated 'gap' between the educational sector and the trades and industries did not appear."

At the time of publication, the math-association addressed itself to the Danish Employers' Association, who had a clear interest in supporting the youngsters' work with science and mathematics. The Danish Employers' Association offered financial support to the publication and promised to be instrumental in facilitating contacts between the trades and industries and interested senior schools.

3. THE CONTENTS OF THE PUBLICATIONS.

The first example in the book shows the construction of a growth model for trout in fish ponds. The optimal way of feeding the fish is found by constructing a growth model. Computers are important here.

The next example goes through some calculations in connection with filtering smoke from refuse disposal plants. Analytic functions are found to fit functions known from tables.

The third example demonstrates the application of statistical methods in process control.

The fourth example goes through the calculations used in dimensioning a system for the production of acetyle-acetone.

The Distillation-publication tries

- to describe the mathematical models underlying distillation
- to give examples of practical, industrial applications of distillation

It is shown how upper secondary mathematics is directly used in the trades and industries. There are two main target groups:

- Upper secondary pupils with special interest in applied mathematics
- Employees in the industries with highschool diplomas but without academic education. The material described is used in the companies for in-service education.

All upper secondary math-teachers in Denmark were given a specimen of the books, and it is now part of the narrative material which is so essential for all teachers. This is true also about math teachers.

4. ONE EXAMPLE FROM THE BOOK, *AUTHENTIC APPLICATIONS OF MATHEMATICS.*

Cheminova Agro is a chemical factory, which intended to make a pilot plant to produce 2,4-pentanedione from acetone. 2,4-Pentanedione is used as an intermediate product for producing medicines and plastic materials.

Development and construction of a chemical production have different phases:

1. The process is investigated in the research laboratory to find a proper method of production. This method is optimized according to profit, improvement of processes etc.

2. Construction of a pilot plant on the basis of the investigations in the laboratory. The aim is to test the process on a larger scale – e.g. 100 to 1000 times the laboratory scale. Here it will become known if problems of a technical or chemical kind arise because of the upscaling. The quality can be tested and the market can be surveyed.

3. Construction of the proper production plant on the basis of the experience from the pilot plant.

The time horizon for the implementation of a project from the laboratory to a proper production plant varies from 5 to 10 years.

To illustrate how mathematics is applied to dimension a pilot plant on the basis of the laboratory-work, we will look at the following problem:

> A pilot plant is to be built for producing 2,4-Pentanedione from acetone. The capacity of production wanted is that 100 kg of acetone are used every hour.

Many different processes must be dealt with and many parameters are involved in the dimensioning of the plant. To get the impression that this complexity can be governed by mathematical models is in itself of great value, i.e. the strength of mathematical models is emphasized.

The mathematical preparations were made for the construction, but for various reasons the plant was never built. Therefore the material is open, and we can use it to illustrate the calculations used in the dimensioning problem.

As a part of the process, you have to change acetone from a liquid at 20°C to vapour at the boiling temperature of 56.2°C.

The specific heat capacity and the heat of evaporation is known, so it is easy to calculate the power needed.

The energy transportation takes place in a circulation evaporizer as sketched in the figure:

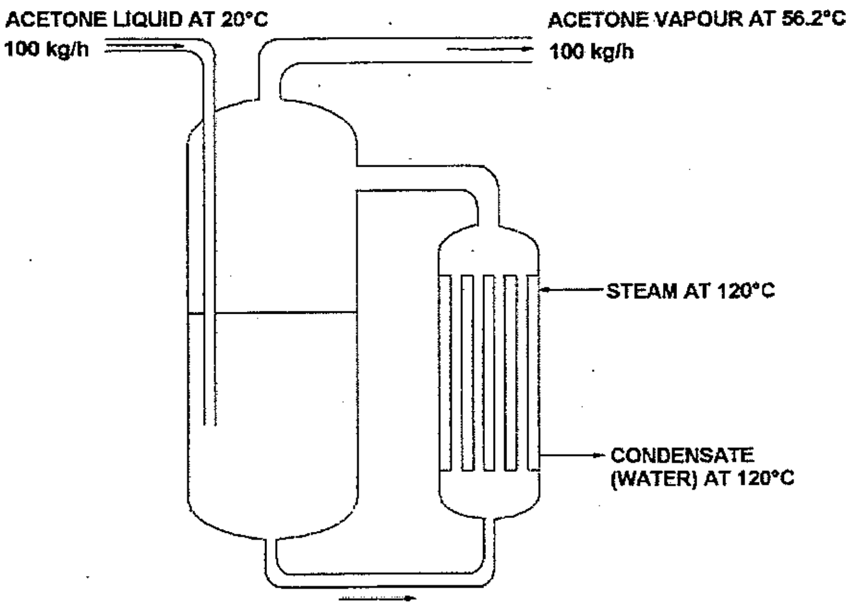

Fig. 1. Circulation Evaporator

As an energy supplier steam at 120°C is used, which is condensed under pressure to water at 120°C in a heat exchanger. Knowing the heat of evaporisation and the power needed, the necessary steam flow can be evaluated.

Typically the flow of acetone through the heat exchanger is made ten times as big as the basic acetone flow.

The main problem is to find the surface area of the heat exchanger, knowing the heat transmission properties of the material $(kJ/m^2/h/K)$ and the acetone flow.

It is easy to find the temperature in the tank which is a mixture 1:10 from different known temperatures.

Which temperature difference in the heat exchanger are we to use? The outer temperature is constantly 120°C, but the temperature of the flow is rising because of the heating.

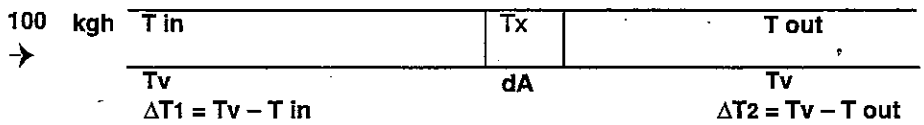

Fig. 2. Heat Exchanger

By comparing the total energy transport with a differential energy transport it is shown that as mean temperature difference you must use the formula for the logaritmic mean temperature difference, $LMTD$:

$$LMTD = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1/\Delta T_2)}$$

where ΔT_1 and ΔT_2 are the temperature differences in the beginning and in the end of the heat exchanger. Hereafter it is easy to find the area of the heat exchanger.

It is interesting to show that if the temperature differences do not differ very much, you can use the expected formula for the mean temperature difference $(\Delta T_2 + \Delta T_1)/2$. In the book it is shown by example that knowledge of the LMTD is necessary.

Here you may break off the exposition, but the book goes one step further.

The acetone is now in vapour form, and the next task is to dimension a pipeline, where the vapour can be further heated and react to ketene. This is done in the book.

5. LEARNING AND TEACHING OF THE MODELS IN THE CLASSROOM.

The models in the publications contain exercises that can be solved in a traditional way. If the students work individually or in groups the process is made easier because of the appendix with suggestions for solving the exercises and results.

Having worked with "Authentic Applications of Mathematics", it is obvious that the students react in a very positive way to these alternative teaching methods.

Their respect for and interest in the trades and industries is increased. This is equally true of their respect for and interest in mathematics since the models clearly show that the very complicated processes of production only can be controlled by mathematical models which among other things show their strength in splitting complex ways of thinking into more simple ways.

It is the teacher's impression that working with applications help further the interest in the trades and industries so that many students thus get a wider selection of future occupations.

Many of the processes in the publication cannot be recreated in the class room which gives ample opportunity for field excursions to the companies.

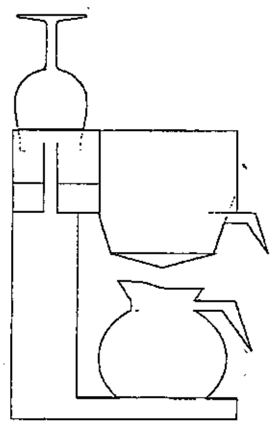

Fig. 3. Model of Circulation Evaporator (Fig. 1.)

On the other hand the processes can often be perceived with a bit of ingenuity. This happens in Fig. 3. where the circulation evaporator of Fig. 1. is illustrated very well by a coffee machine and a wine glass.

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Hermann K. and Hirsberg B. (1991). The Ministry of Education, Denmark, Mathematical Models and Modelling. Examples from Danish Upper Secondary Teaching. In: Blum W., Niss M., Huntley I. (eds.), *Teaching of Mathematical Modelling and Applications.* Chichester: Ellis Horwood.

Touborg, J. P. (ed.) (1991), *Autentiske matematikanvendelser*, København: Matematiklærerforeningen.

15

Partnership Modelling between Industry and University

Michael Hamson
Glasgow Caledonian University, UK

SUMMARY

An MSc course in Industrial Mathematics is in operation in Glasgow, mounted jointly by the Departments of Mathematics at the University of Strathclyde and Glasgow Caledonian University. It is intended to provide generalists in applied mathematics for careers in industry and business. The year long course consists of taught modules, team projects and an industrial placement.

This Paper describes the *Team Project* activity in which groups of students work on problems usually supplied by industrial contacts. The duration of each project is strictly monitored and students report on the outcome by written and oral presentation. In this way it is hoped that the project work reflects industrial practice to the subsequent benefit of the students.

The Paper describes how the *Team Projects* are organised and assessed and gives details of the activity of one particular project carried out in January 1993.

1. BACKGROUND

The Course Leaders of the Glasgow M.Sc. programme, taking advice from many industrial contacts, are of the opinion that vocational postgraduate work in mathematics is not so much specialist but generalist in nature. There is a need for the young professional mathematician in industry to be versatile in applying his/her knowledge of the subject, be prepared to work in small teams on timed projects and be able to communicate the results with others who may not be mathematicians. This communication would be a report delivered probably to a manager whose interests are in decision and policy as opposed to mathematical detail. While mathematical rigour would of course have been established and checked earlier, a manager would need convincing of some suggested new course of action. Learning team activity skills of this kind are not usually a part of other specialised M.Sc. courses dedicated to one particular branch of mathematics.

Further, industrial mathematicians perhaps working in research and development, can be asked to carry out computer simulations through constructing mathematical models. This will probably be in order to save money compared with live trials, or where a trial is impossible (say an environmental issue or nuclear waste disposal). Model building is at the heart of the Glasgow M.Sc. Most students joining the course have no previous experience of constructing their own models – building one's own mathematical model is of course quite different from using someone else's model.

Based on the above objectives, the curriculum for the M.Sc. is drawn up to contain a blend of subject module taught material, team modelling projects and an industrial placement. This would seem to be a fairly unique type of Masters course in the U.K. With the aim of producing generalist skills, the subject modules include a wide range of topics: fluid dynamics, signal processing, experimental design, time series and forecasting, elasticity, codes and ciphers, heat and mass transfer and so on as well as foundation work in statistics, operational research, numerical and mathematical methods, optimisation, scientific computing and software development.

The criticism often made of raw mathematics graduates employed in industry is that they take some time before they can usefully tackle real problems. The best postgraduates are usually very well equipped for dealing with standard theoretical and classical applied mathematics but cannot easily convert this knowledge productively onto the actual industrial scene. As has been stated above other skills are wanted: awareness of who the 'customer' is, and the ability to discuss a provided problem with an engineer or business manager. Within the Glasgow course, team project work, and later the industrial placement, help to provide this skill and also set up a *partnership* with industry who provide much of the material for the projects. This is where students learn the rough side of the business; some do not take kindly to perhaps ill-formed problems and to team work debate, perhaps leading to criticism from fellow students.

2. TEAM PROJECT ORGANISATION

A bank of modelling problems are assembled from contacts with industrialists and these are added to by 'learner projects' supplied in-house. All students take three such projects and work in teams of three with one person designated as team leader. The leader has the responsibility to liaise with the academic supervisor and to organise the modelling work within the team. This person is separately assessed for the leadership quality. Two of the team projects normally run over a five week period of time alongside taught modules, but the third is concentrated over two weeks without interruption, with stricter deadlines. For all team projects, oral presentations are held and written reports produced. The resulting outcome is discussed with the industrial problem provider if possible. Sometimes an industrialist will come to the Universities to present team project material directly to students.

Objectives for Team Projects can be summarised as follows:

As a result of taking part students can

(i) time manage their work better,

(ii) learn how to work productively as members of a small team,

(iii) tackle open-ended real problems by formulating mathematical models and intrepret answers back in terms of the original problem,

(iv) take technical decisions necessary in the model building exercise,

(v) communicate the outcome in oral and written form,

(vi) react positively to peer group pressure and assessment,

(vii) produce support computer software and use a range of packages as necessary,

(viii) prioritise tasks with different time scales,

(ix) realise that more than mathematical techniques are needed to solve a model–consultation and planning are needed first so that the model is correctly set up,

(x) deal constructively with feedback.

A very good preparation is established from all this for the four month industrial placement that follows for all students on the Glasgow course. The partnership established can hopefully be extended during the placement since quite often this takes place at the same Companies who have provided the earlier team projects.

Assessment of Team Projects:

As can be imagined this is often a vexed question for all modelling work no matter what level. Plenty has been reported on this elsewhere. In this situation, care is taken to note team progress and leadership effectiveness during a project and also the amount of help necessary. Three assessments are given for each project:

(i) technical development,
(regular meetings between the team and academic supervisor, with contact to the industrial problem provider if necessary)

(ii) oral presentation,
(each student will describe (in 7/8 minutes) a part of the project and will be questioned on this)

(iii) written report.
(to be of length about 25 A4 pages; each student must insert a short paragraph stating which part of the total project work they are responsible for).

Individual marks are awarded for (i) and (ii) but a team mark is given for the written report.

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(i) time manage their work better,

(ii) learn how to work productively as members of a small team,

(iii) tackle open-ended real problems by formulating mathematical models and intrepret answers back in terms of the original problem,

(iv) take technical decisions necessary in the model building exercise, These assessments are judged by the academic supervisor and a second assessor and, where appropriate, the industrial contact helps with this. Feedback takes place on all the team projects in which the supervisor interviews each team member separately to confirm knowledge and contribution in the project. This is also necessary before awarding the leadership mark. Team members have the opportunity to comment on the effectiveness of the team leader and whether the project proceeded harmoniously.

Care has been taken by the Course Organisers to ensure that clear guidelines are used in the assessing of presentations and written reports. Students are given advice in the preparation of both these communication skills at the start of the course. Reports are expected to be Wordprocessed and drawn up in a manner consistent with standard technical reporting: to contain summary, contents page, main body, conclusions, appendices, references etc. As far as oral presentations are concerned, overhead projectors and/or flip charts are used and students are expected to share the delivery equitably among their team and to communicate coherently and be able to deal with questions.

About one fifth of the course is taken up with team projects; some students carry out this activity in Europe as part of the Erasmus interchange scheme. (Others are selected to attend the ECMI modelling week [1]).

3. RECENT EXAMPLES OF TEAM PROJECTS

1. 'Hall Effect Current Sensor'
   –Honeywell Control Systems, Motherwell, Scotland. [see 4. below]

2. 'Transport of Microbes through a Porous Media'
   –British Nuclear Fuels, Risley, Warrington, England.

3. 'Pipeline Gas Network Compressor Efficiency' –British Gas, Research and Technology, Newcastle, England.

4. 'Vehicle Routing in Milk Collection'
   –Milk Marketing Board, Surrey, England.

5. 'Measurement of Railway Performance'
   –Scotrail Glasgow, Scotland.

6. 'Aircraft Response in Turbulence'
   –Defence Research Agency, Bedford, England.

7. 'Theory of Viscometry at Elevated Pressure' –National Engineering Laboratory, East Kilbride, Scotland.

Some of the in-house projects:

8. 'Traffic Light Phasing in Glasgow City Centre'

9. 'Asset-liability Mismatching by Insurance Companies'

10. 'Modelling Epidemics'.

4. PROJECT: HALL EFFECT CURRENT SENSORS

Problem Supplied by Honeywell Control Systems, Motherwell, Scotland

The Honeywell company manufacture microswitches and processors used in electronic household appliances. One such device is a current sensor based on the null balance principle. An annular steel core has a magnetic field induced in it by a current carrying conductor placed through its centre. This field is opposed by another generated by passing a current through a coil wound onto the core. The objective is that the two fields cancel to provide a 'null balance effect' which is detected by a 'Hall element' embedded in the core. The device is shown in Fig.1. below.

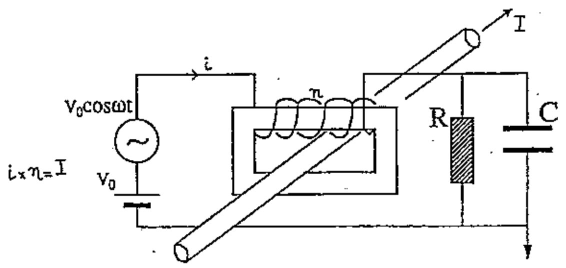

Fig. 1

The current in the coil is also an indicator of the behaviour of the system, and the voltage drop producing this current satisfies the differential equation:

$$\frac{d^2v}{dt^2} + k\frac{dv}{dt} + f(v) = b_0 + b_1 \cos(2t) \quad (1)$$

where v is the voltage at time t,
k is a damping coefficient,
b_0 and b_1 are forcing coefficients,
and $f(v)$ takes the nonlinear form $av^3 + (1-a)v$.

It was thought that critical ranges for the parameters are

$$k \simeq 0.1 \,,\; 0.1 < a < 0.4 \,,\; 0 < b_0 < 2.0 \text{ and } 0 < b_1 < 5.0.$$

The behaviour of the voltage is required taking into account the parameter variability.

Initiation of the Project.

The problem provider from Honeywell has a general mathematical services role to support scientists and engineers. There is an element of the 'all-rounder' about this sort of industrial mathematician as advice is sought from this person within the Company on a wide variety of problems. The students who chose to take up this project had no previous experience of microswitches and current sensors (though a Course visit had been made to Honeywell earlier).

The Course Team normally encourages the students to form their own teams and then to select from a range of projects offered which will normally be more in number than is required. When a project has been especially provided from an industrial contact, then usually a team will be directed onto this, perhaps leaving in-house projects as the spares. In this case the team was acquainted with the project directly before the Christmas break and then returned in early January (1993) to work on it over a short two-week period.

The project for the M.Sc. students centred around obtaining some feel for the behaviour of the voltage $v(t)$ in (1) above for the parameter values suggested. The major demand on the team was caused by the nonlinear effect of $f(v)$. The Company interest in this problem concerned what happens when the grade of the steel used in the core is changed for cost cutting reasons. The effect of this was to produce unwelcome low frequency behaviour in the voltage known as sub-harmonics. With regard to (1), the model of the switch sensor, this means we are looking for a solution which exhibits periodicity of lower frequency than that generated by the forcing terms. The resulting 'slow' oscillations, perhaps remaining un-suppressed by damping, are of concern to the Honeywell engineers.

Team Procedure

The student team had received preliminary information when selecting the project. On their start day a detailed 'FAX' was waiting from Honeywell giving fuller information. They were not expected to research the background of current sensors and so were presented with a well-defined problem.

Despite possessing sound knowledge of linear differential equations, equation (1) needed the team to try more advanced methods in its investigation. Reference [2] seemed a good source. There was a dilemma whether to concentrate on analytical methods detailed in [2] or whether to make major use of a numerical differential equation solver, package [3] being available.

It was decided that it would simplify the analysis at the outset if t were replaced by $\tau/2$ to re-scale (1) as

$$4\frac{d^2v}{d\tau^2} + 2k\frac{dv}{d\tau} + \varepsilon v^3 + (1-\varepsilon)v = b_0 + b_1 \cos\ \tau \tag{2}$$

where the parameter a has also been replaced by ε, to indicate that it is to be regarded as a small quantity so that (2) can be then be tackled by the perturbation methods. Now it can be seen that (2) has a forcing term with period 2π, so that the subharmonic behaviour that was being sought would be manifested by solution terms of the form $\cos\ (\tau/2), \cos\ (\tau/3), \ldots$, known as sub-harmonics of order 2,3,.... and so on to produce periodicity $4\pi, 6\pi, \ldots$.

The perturbation method means a solution of (2) is required as an expansion of $v(\tau)$ in powers of ε in the form

$$v(\tau) = v_0(\tau) + \varepsilon v_1(\tau) + \varepsilon^2 v_2(\tau) + \ldots$$

When coefficients of ε are equated, a series of LINEAR differential equations result:

$$v_0'' + 2kv_0' + v_0 = b_0 + b_1 \cos\ \tau \tag{3}$$

$$4v_1'' + 2kv_1' + v_1 = v_0 - v_0^3,\ \text{etc.}, \tag{4}$$

From the first of these equations it is clear from the form of the left hand-side that solutions involving cos $(\tau/2)$ *must* occur which would persist for small damping. Search for terms in cos $(\tau/3)$ proved more difficult. The approach favoured by the students was first to neglect the damping term, reasoning that damping would be a mechanism for suppressing the subharmonic effects. A solution of (3) with $k = 0$, may be taken in the form

$$v_0(\tau) = b_0 - (b_1/3)\cos\ \tau + A\cos\ (\tau/2), \quad A \text{ arbitrary.}$$

Hence the right hand side of (4) may be evaluated and provides, after suitable rearrangement, a series of cosine terms:

$$c_1 + c_2 \cos\ (\tau/2) + c_3 \cos\ (\tau) + c_4 \cos\ (3\tau/2) + \ldots + c_7 \cos\ (3\tau), \tag{5}$$

where the coefficients c_i depend on the earlier parameters $b_0; b_1$ and A.

While (4) is *linear*, obtaining its solution is not so useful as appraising its right-hand side. The students turned to [2] for help and from Chap. 7 came the comment that 'a subharmonic of order three is stimulated by an applied frequency three times the natural frequency of the original linearised equation'. Thus the term $c_4 \cos\ (3\tau/2)$ in (5) seemed to indicate the presence in $v_1(\tau)$ of the required subharmonic, while other terms perhaps indicate subharmonics of higher order as well. The size of the c_4 coefficient would determine the 'visibility' of the subharmonic and this was tied up in the values of the parameters b_0, b_1 and A.

The students were able to report on this outcome, but noted that its validity depended on the legitimacy of the perturbation method with sufficiently small ε. They were unable in the time available to find parameter values which would deliver the subharmonic of order three. It was decided to make use of the numerical solver package [3] to see if the behaviour could be shown graphically. A number of trials were attempted in obtaining the approximate solution of (1), with $k = 0$ and taking various choices of a, b_0 and b_1. A subharmonic feature was found present in most of the graphical output, but not of order three. The most clear cut evidence achieved in the time available is shown on Fig. 2.

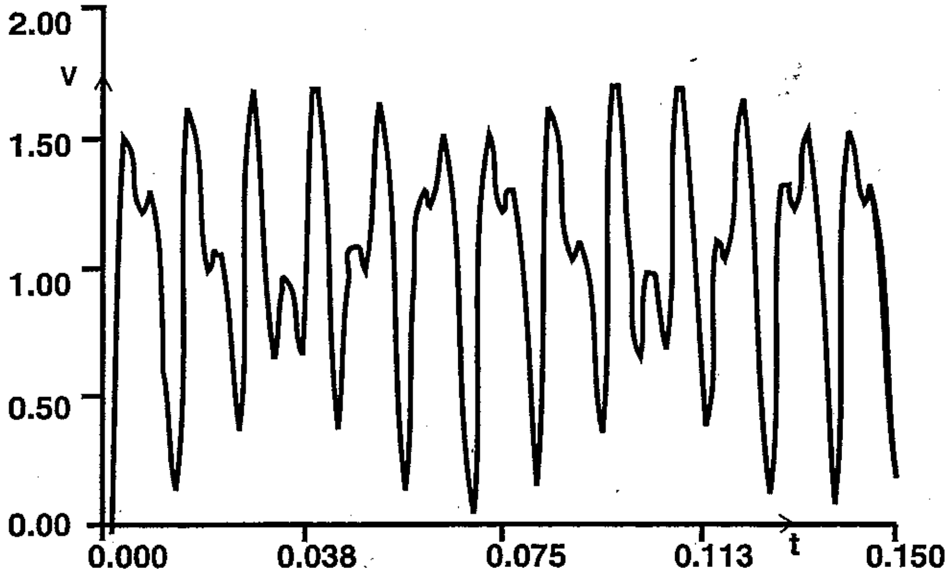

Fig. 2

Here as the cubic term becomes small the outcome will revert to the linear solution:

$$v = 1 - (1/3)\cos\ \tau - n(2/3)\cos\ (\tau/2),\ \text{period } 4\pi.$$

Team Project Outcome

The interim results indicated above were reported. The parameter variability needed some confirmation from Honeywell before the subharmonic effect could be fully understood. The Company viewed the students' effort as a useful preliminary investigation to a more thorough treatment needed. The problem provider at Honeywell was more inclined to trust analytic methods compared to numerical output generated from a differential equation solver package. He suggested the project should be attempted over a longer period of time so that more work could be devoted to finding the order three subharmonic. The achievements of the team had been satisfactory, although on *feedback* between supervisor and students, their knowledge of the general behaviour of differential equations was found to be somewhat limited, confined mainly to a good grasp of classic solution techniques.

5. CONCLUSIONS

Notwithstanding the incompleteness of the example given in 4. above, there was a lot gained from this work. Considerable insight into micro electronics, non-linear differential equations and general harmonic analysis resulted. There were also lengthy discussions with the Honeywell company. The activities give some indication of how *partnership projects* work on the Glasgow M.Sc. There are often unresolved outcomes giving rise to further industrial projects for later use. Nevertheless there can be some dangers in taking raw problems for direct use with students when the time needed to obtain a reasonable solution cannot be forecast accurately. With this particular project the students suffered some negative feedback since they were not especially successful with the problem and the Company clearly required and expected more work to be done. Tutors at Glasgow Caledonian University have since been tackling the current sensor problem themselves! Generally however, team projects supplied by industrial partners have been most welcome and have helped students prepare for the real world of industrial mathematics.

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16

Scope of Mathematics Practitioner Involvement in Undergraduate Mathematics

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SUMMARY

This paper outlines a compulsory second level mathematics module at CGCHE.

Maths is a main field in the college's undergraduate modular degree scheme. One of the aims of the module is to develop an appreciation of the role of the mathematician in commerce and industry. The involvement of such practitioners in the module is discussed and a number of problems set by them are presented. The incorporation of these problems into an assessment pattern is also discussed.

1. INTRODUCTION

At the above college, there are currently 90 students studying mathematical modules at the second level. 70% of these students are studying for a B. Ed. Degree (Primary) and when qualified will teach children up to the age of 11. These students will complete their study of mathematics at the end of level 2. In order that all students have a broader view of mathematics and mathematicians, the module entitled "Scope of Mathematics" has been devised with the following aims: Students should develop an appreciation of:

i. the role of mathematicians in industry,

ii. the possibility of mathematical analysis of recreational activities,

iii. the fascination of abstract mathematics.

The course is delivered by a team of tutors through informal lectures, tutorial sessions, a Mathematicians in Industry Conference and student presentations.

Topics presented by the staff through informal lectures have included: Recreational Mathematics, Number Theory and Ciphers, Population Modelling, Music and Mathematics, Psychology and Mathematics, Logic and Proof.

This section of the module will operate for the first seven weeks.

2. MATHEMATICIANS IN INDUSTRY CONFERENCE

This session, which runs for three hours, has up to 12 "practitioners" drawn from industry and commerce. The practitioners reflect the nature of local employment in the region which has a number of light engineering companies, several connected with the defence industry, and also a number of insurance companies and building societies have their headquarters in Gloucestershire. The focus of the day is to look at the role of the mathematician and not solely at the mathematics. Students have been advised to prepare their own questions prior to the conference. On the day the students are divided into groups of 7 to 10. Students and a practitioner begin with a groupwork activity on The Indivisible Load (Industrial Society 1990) which has been widely used by the Industrial Society as an Icebreaker activity. The students then spend 30 minutes with the practitioner discussing the practitioner's role in industry. The practitioner then moves onto another group of students for a second witness session. The final 30 minutes of the conference consists of introductions from a number of practitioners to the problems and case studies they have devised. Students will choose one problem to investigate and present as a group project at a later time. This is part of the assessment pattern for the module. Some examples of starting points for projects are presented in the appendix.

3. RATIONALE

The above pattern has been constructed to provide opportunities for industrialists to communicate with students in an effective manner. One-off large group lectures are not ideal ways of communicating ideas and would be seen as intimidating to an employee not used to presentation. The question arises as to whether it is realistic to try to give students a feel for industrial mathematics in the time available. Edwards (1991) comments that the development from graduate mathematician to industrial mathematician takes time and that the proportion of time spent in those activities instantly recognisable as mathematics is relatively small. The two mathematical attributes he considers to be of immense value are the confidence to ask "why"? and the confidence to say "I do not understand" which are central to the spirit of the Mathematics Field of the college. It is important for practitioners and students to explore their commonality of belief in a situation that allows such exchanges to occur.

4. EVALUATION OF THE MATHEMATICS IN INDUSTRY CONFERENCE

The module has operated for three years and at the end of each conference an evaluation form was completed by all students (in pairs). The form was intended to give students a fairly open-ended opportunity to comment. Analysis of these questionnaires indicated that the format of the conference has met with general approval from students, who welcomed the opportunity to ask questions of practitioners. Frequently occurring phases on questionnaires were "worthwhile, interesting, informative, useful, helpful". Students' views on the witness sessions naturally depended, to a certain extent, on whom they spoke to but generally it was recognised that there was a good variety of practitioners with differing occupations and backgrounds. A conscious effort had been made to invite a balanced proportion of female practitioners which was commented on and appreciated by a number of students. Students found it useful to hear about some of the possible uses for a mathematics degree.

Perhaps the most revealing comments came from the *students* whose images of mathematicians had changed, usually as a result of realising the variety of applications of mathematics. We received comments such as the following

"You do not have to be a mathematical genius to get ahead in industry."
"Broader range of mathematics used than previously thought."
"We discovered that maths is not always directly involved and that it is not just a question of solving piles of sums."

The opinions of those students who were intending to become teachers were specifically sought and how they might apply what had been learned from the course and the conference in particular. As might be expected there was a wide variety of responses. The following give some idea of the range of comments:

"Try to learn more about industry so I can relate it to my teaching."
"Stress the importance of teamwork, communication etc."
"Inform children a little more about industry, initiative and enterprise."
"Don't paint a bad picture of industry, it is not that horrible."
"Can I really apply it? It just made me more aware of certain aspects of my own and other personalities."

Some students felt that they should have had a free choice of practitioners and on the negative side, however, one response stated "sadly confirmed, more statistics and computers". The views of the practitioners were also obtained and they universally welcomed the opportunity to take part in the conferences. Comments expressed were:

"As a forum for the practitioners it was probably a rare experience...an unusual exchange of opinions."
"A pleasant change from a morning at work."

Practitioners also remarked on the range of background of students:

"There was a wide range of mathematical backgrounds, consequently a wide range of interests needed to be catered for which was difficult."

The role of group work and the use of the icebreaker activity received mainly favourable comment:

"From a range of mathematical specialisations the groups communicated well among each other."
"The transport problem (icebreaker) worked...I gave a lot of guidance

at first then they took over."

Several practitioners felt that they wanted more time for the event and some recognised that allowing students a choice of topic or focusing on those wishing to follow a mathematics based career were possible improvements while noting, however, the potential problems with these approaches. When asked whether their view of Mathematics in Higher Education had changed, the practitioners highlighted the strengths and challenges of the Modular Scheme:

"Very practical and tuned in to what is required of them when they leave college."
"Seems to be a lot more choice nowadays, though this probably reflects my background."

5. ASSESSMENT OF THE MODULE

Assessment is by means of one assignment and one presentation. The assignment consists of a portfolio of work reflecting the topics covered in the first part of the module. Students work in pairs and they use "starter problems" provided by the tutors as a basis of developing and extending their knowledge. Evidence of an ability to pose and answer their own questions on topic areas, as well as extensions to the starters, is sought. A clear analysis of how the pair worked together e.g. the nature of individual contributions is required.

The second component of assessment is a presentation by a group of three or four students of a project chosen after the conference. The presentation is made in front of a subset of students and two tutors who will award the group a grade. The group may then share out credit within certain constraints. Prior to the presentation a group will decide criteria and weightings for each criteria. An agreed peer assessment rating sheet is then handed to a tutor. Students will not be graded at this point but all students will be aware of how internal assessment is conducted. A tutor will act as arbiter if a group cannot agree on an individual's score and the tutor's decision will be final. An exemplar of how the system works is given below (see Gibbs *et al*, 1989).

MT 201 PEER ASSESSMENT RATING

- - - has contributed to the groups work in the following way

	CONTRIBUTION		
	MAJOR	AVG.	MINOR
1. Organization & Management	+5√	0	−5
2. Ideas & Suggestions	+3	0√	3
3. Data/Evidence Collection	+3	0	−3√
4. Analysis of Results	+3	0√	−3
5. Proofs or Conclusions	+3√	0	−3
6. Report Writing	+3√	0	−3
TOTALS	+11	0	−3
Group Mark to be Entered by Turor	A	60	
Individual Students Total Rating	B	8	
Group's Avg. Total Rating	C	+1.2	
FINAL GRADE	(A+B-C)	67	

We have adopted the approach of allocating different marks to the different members of a group in order that their relative contributions are reflected. We feel that it offers a solution to the problems associated with assessment of group project work, for example unfairness to individuals, high average marks and narrow bands of marks. The criteria and size of awards/penalties can be negotiated with the students or even be determined by them at the start of the project so that they are aware of how they will be assessed and have a commitment to the criteria.

Our observations are that students appreciate the opportunity to contribute to their assessment pattern and try to show respect for other colleague's ideas and commitments. The tutors have had to arbitrate on the individuals scores within a group on just one occasion and this was settled amicably.

6. CONCLUSIONS

The experience gained at Cheltenham and Gloucester College is such that the nature of this conference and the subsequent project work in groups has been beneficial to the students. Students and practitioners have been comfortable with each others' role at the conferences and have been prepared to make positive contributions to the events. It is important to chose practitioners who are interested in the students, their backgrounds and motivation for studying what they do. Choice of practitioners was made initially through personal contact with mathematicians in the centre and this has led us to a network of interested parties and contacts with further companies in the locality. The practitioners involved have become familiar with developments in higher education in the college, particularly with the scope and objectives of the modular degree.

We would hope that the prospects of employment for our students after graduation have been enhanced with these contacts. For those students who will become teachers, we feel that benefits will accrue for both these people and the children they will eventually teach because the impression that industry and commerce is an alien and hostile environment will not be propagated. The important aspects of relevance to vocational areas and the need to develop common skills and realism, in terms of situations, contexts and tasks involved in the assignments and contacts with interested parties, have clearly been exhibited. Some students have been motivated by contact with the mathematicians from industry and commerce to pursue the topics introduced in the conference as projects or investigations at a later stage of their degree. For instance, the project on image functions (see appendix) has motivated one student to investigate in greater depth the field of image processing and he is now in active collaboration with the practitioner from the company concerned. We will continue to monitor the development of the conferences. For instance, some issues which will need further considerations for future events are:

i) the length of each witness session,

ii) use of the mature students' experience,

iii) student choice of practitioner sessions and the possible involvement of mathematicians who have transferred from industry to teach in schools.

We would recommend that other institutions actively consider this method of investigating mathematics in industry. By discussing the role of the mathematician rather than the mathematics itself, a realistic impression may be gained by the student. It is clear that dialogues between practitioners and students can eliminate misconceptions on both sides. However, we recognise that without the good will and enthusiasm of practitioners our conferences would not have been effective at changing attitudes.

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APPENDIX

The following are a sample of problem statements as set by the practitioners at the Mathematicians in Industry conference. Where appropriate, however, relatively confidential information has been omitted. Practitioner problems have differing levels of authenticity. For the student they all represent a higher level of authenticity because of the context in which they are represented. We hope the list of problems below illustrate the wide variety of areas from which our practitioners have been drawn.

A PUMP PROBLEM

A pump's performance may be represented by three curves, Produced Head against Flow Rate, Absorbed Power against Flow Rate, Efficiency against Flow Rate. When a pump is being modelled mathematically during the design process, the "Best Efficiency Point" (BEP) is first fixed in order to specify the optimum flow, head, efficiency and consequent power absorption. In addition to this, the head may be specified at two additional points, one at zero flow and another, for example, 140% of BEP flow. Based on given figures, find an equation of a smooth curve for which the equation is known.

The efficiency will also be represented by a curve which is fixed by three conditions : its value is 0 at zero flow, its value at BEP is known, it must peak at BEP. Deduce a method of meeting these three efficiency conditions with another smooth curve for which the equation can be found. Given that the Head-Flow and Efficiency-Flow curves are now defined in equation form, can the power be represented as another smooth curve for which an equation can be found? Hence find the power which is absorbed at intervals of 5 l/s between zero flow and 140% of BEP flow. What effect would there be on the modelling process if additional efficiency points are specified? The Head-Flow curve of a pump is often best represented by a line at low flow, becoming a curve at higher flow. How would this affect the modelling process?

A PUBLISHER'S PROBLEM

What advice is to be given to authors on format (page size) and extent (numbers of pages), possible use of photographs and a second printing colour in order to achieve an acceptable profit on a "reasonable" published price with sensible print runs over four years for a new mathematics text book.

Data provided to students included: two formats, print-run parameters, pricing parameters, costing formula and parameters and analysis of costs.

IMAGE FUNCTIONS

An image can be described by a step function in two dimensions. The object of this exercise is to reduce the data required but still have a fairly accurate description of the image. To simplify matters take a one dimensional cross section of a tiny part of the image. You might get a function such as lm below. lm could be described by an ordered set of numbers: $-lm = \{20, 16, 10, 12\}$

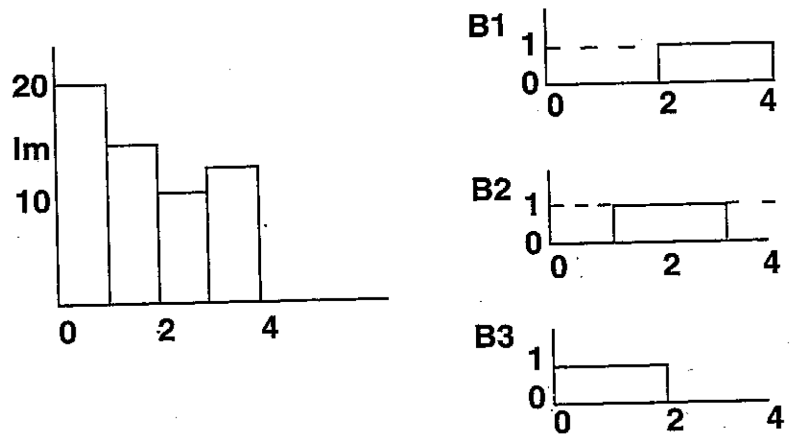

Fig. 1

Suppose now we invent 3 base functions such as B1, B2, and B3 above. They could be described as $B1 = \{0,0,1,1\}$ $B2 = \{0,1,1,0\}$ $B3 = \{1,1,0,0\}$ We could make an approximation to lm by combining $B1$, $B2$ and $B3$ in the right quantities. For example, the function P below gives a fair approximation.

$$P = 13B1 - 4B2 + 20B3.$$

Calculation shows that $P = \{20, 16, 9, 13\}$

We could describe P by just 3 numbers [13, -4, 20] and hence we have a small saving of data.

Now you might like to consider the following problems :-

1. Is the approximation above the best we could get with the 3 functions defined? What is a good measure of "best approximation"?

3. Is there a good way to calculate the best possible combination of the functions?

3. Was it a good choice of base functions? Could you choose functions that would be easier to calculate or which would make it easier to predict how good the approximation would be?

4. Could you generalise the methods to deal with functions defined in 8 intervals?

PERMANENT STAFF VS. OVERTIME

You have to determine the optimum number of staff within a central warehouse function so that the annual wage costs are a minimum. This is to be undertaken for a period of three years. Students are presented with rules and basic information such as hourly rates of pay, union agreements etc. and the profile of the work throughout the year.

MAXIMISING THE EARNINGS OF AN EXTRUSION PLANT

One of the major activities of a particular company is the production of aluminium extrusions for customers to subsequently convert into aluminium products such as windows, doors etc. The management wish to investigate the effect of different volumes of extrusions of different film thickness being processed in its anodising plant.

Students are presented with information and numerical details of the anodising process, pre-treatments, post-treatments and earnings for the various film thickness.

SCHEDULING THE PRODUCTION OF TEXT BOOKS

Consider how to schedule the manuscript stage prior to editorial/production and how to arrive at a pre-designated publication date, 21 months after the whole manuscript cycle starts. This cycle requires a minimum of 36 weeks. Students are provided with information relating to the edit/production schedule.

Section E

Primary and Secondary Examples

17

The Value of a Modelling View on Primary Children's Problem Solving

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SUMMARY

This paper examines the value of using a modelling approach to analyse children's practical mathematical activities, in order to interpret their experiences and to develop their problem solving skills. We will relate the content of our college course to our subsequent classroom teaching by drawing parallels between the skills and attitudes which we encountered at our level, and those that we observed to be developing in young children while involved in structured Mathematical Modelling activities.

1. A COLLEGE BASED INTRODUCTION TO MATHEMATICAL MODELLING

We specialised in mathematics as part of our four year B. Ed. degree course. In the first year, studies were carried out at our level while in the fourth year the content was applied to the primary age range. Following our initial introduction to Mathematical Modelling, a formal lecture, we were presented with a real life problem to solve using the established principles. This involved examining a situation whereby two trains were required to travel along the same length of track at similar time, and making recommendations for a time-table which avoided any collisions or delays.

In the final year of our college course we were again involved in Mathematical Modelling activities, this time with junior age children. This project required the ten-year-olds to examine the most efficient way of policing a football match, using a computer simulation called PLOD. We observed that the children were able to indirectly make use of the Mathematical Modelling sub-skills to solve the problem. Realising that Mathematical Modelling could play an important part in children's mathematical development, we were stimulated, as early years specialists, to consider whether it would be possible to do such work with infant age children (4-6 year olds in our case). It seemed a fundamental error of judgement to deny them these experiences simply because of their age, as so often happens.

As infant teachers we realise the value of structured play activities in child development and found this practical approach conducive to Mathematical Modelling activities. The ability of young children to become involved in such activities is illustrated by the following examples.

2. SIMULATED HOUSE CONSTRUCTION IN A RECEPTION CLASS

While working with a Reception class of five-year-olds on the theme of 'Homes and Families', a number of children decided that they would like a 'real house' in the classroom, rather than being content with the home corner area. We discussed the possible building materials that could be used. The children eventually agreed to use old cardboard boxes when they realised that there was a limit to the amount of the other construction materials available, such as lego and plasticine. Prior to actually building the house, the children each drew a picture of the house they wanted to build. Over the course of the next two weeks the children discussed their ideas between themselves and with the teacher.

During this period the teacher's role was that of a facilitator, supporting the children and ensuring that their practical needs were met as far as possible. This enabled them to achieve the full potential of their ideas, minimising the frustration which can occur as a result of physical barriers and limited resources. At the end of the two weeks they had successfully constructed a 'real house' within the classroom. It was approximately 1.5 cubic metres in area, with an open doorway, spaces left to represent windows and a triangular roof.

Although the children accepted this as a 'real house', it was obviously a simulation of a real world situation. It is useful to look at the children's activity from a modelling point of view, using the seven sub-skills indicative of such an approach (Shell Centre for Mathematical ed. 1983). These are listed below with relevant examples of the work that related to them.

Generating Variables

This involves considering the factors pertinent to the problem. The children had to decide if the house was to have doors, windows and a roof. While some of these factors were planned ahead, such as the positioning of the door, others were handled on the spot as the children became aware of them, such as labelling any spaces that remained between boxes as windows.

Selecting Relationships

This skill involves an increasing awareness of different variables which relate to each other. For instance, the children discussed how the size of the house was dependent upon the amount of construction material available, and the space available for building related to the number of children likely to play in the house.

Generating Relationships

Having established that a relationship exists between variables, the process of generating involves the identification of these relationships. In the following example the continuous process of selecting and generating relationships is demonstrated. One of the children involved in the activity was able to explain the relationship between a person's height and the height of the door.

Katie: "You won't be able to get in."
Teacher: "Why not?"

Katie: "Because you're too big! I can fit in, but I'll have to duck a bit here. The new little children will fit in though.

The children were satisfied with this simple solution to their problem; small children can walk in, taller children have to duck down, while adults are too tall to go in the house.

Identify Specific Questions

The children had to consider and agree upon solutions to specific questions which emerged during the practical activities. These questions were posed by both teacher and children, such as:

Where is the door going to be?
What shape will the roof be?
Many of these questions involved ideas of shape and space and the application of existing knowledge of building structure.

Modelling

At a basic level this involves the practical application of mathematics to a real life situation. This necessitates making assumptions and simplifying essential features, enabling the application of existing mathematical knowledge to solve the problem. The children discussed having an 'upstairs' during the building process. However, using their knowledge of shape and weight, the children realised that this was impractical and simplified their initial ideas; they built a house with a single room.

Estimating

Estimation skills were constantly in use. It was necessary, for example, to estimate which box was most appropriate for a given space in the wall. During the construction process it was noticeable that the children's estimation skills developed so that they were increasingly able to make more accurate judgements.

Validating

This involves evaluating the result of the modelling process in order to assess its validity and make any necessary adaptions. The children were satisfied with the basic house construction but chose to adopt specific aspects of their house to meet their needs. They made post-boxes and furniture, and improvised a cardboard box as a rotating door, so as to create a more stimulating play area which fulfilled their concepts of a 'real house'.

Inherent in all these sub-skills are the ideas of problem solving and investigation. The development of these is essential because they facilitate the application of mathematics in a variety of situations, thereby making it useful.

While encouraging independent work, the teacher must intervene at opportune moments. It is important to recognise the role of positive intervention in ensuring breadth of experience and continuous development. In such a manner it is possible to provide structure to the experiences without dictating children's activity.

3. INFANT FUND RAISING

An alternative way of analysing children's involvement in problem solving activities is to use the modelling cycle. The work in this example was carried out with a class of six year olds. Formulate the real problem As part of the annual school sponsored event week, it was agreed that each class should decide upon a fund-raising activity. Our problem was to choose the most effective way of raising money. After a session of brain-storming the children decided to bake cakes and sell them in the school tuck shop. This was a popular choice with the children as they already baked on a regular basis with parent volunteers. It was a rare opportunity for them to become involved in a real world problem solving situation, as opposed to the simulations which are commonplace in infant classrooms.

Assumptions Made in the Model

Due to the egocentric nature of infant children's thoughts, it was extremely difficult to get the children to express the assumptions they were making. It was apparent that the children were making automatic assumptions, such as the ability of people to pay for the cakes. The teacher's role was to discuss these implicit assumptions with the children in greater detail. Other assumptions included:

That the ingredients were obtainable;
Parental help was available;
That other children would want to buy the cakes.

Formulate the Mathematical Problem

The children then needed to decide what they wanted to bake. This led on to a discussion regarding which product would be the most successful. Many suggestions arose and eventually the class agreed to conduct a survey of the potential customers. The children had conceived the mathematical problem as being to discover which was the most popular cake, at this stage they had no concept of cost and profit.

Solve the Mathematical Problem

The children had previous experience of using tally charts to record results of surveys, and chose this familiar method in this situation too. In order to ensure a worthwhile survey, the teacher suggested that the children should select three cake categories for the tally chart. They based their choices on previous experiences of baking and were able to conduct the survey independently throughout the whole school, returning to class to discuss their results.

Interpret the Solution

Using the results of the survey the children's overwhelming conclusion was that Rice Krispie cakes were the most popular.

Validate Model

On agreeing that we would bake Rice Krispie cakes, the issue of cost emerged. This was necessary as an essential part of the validation was to ensure that a profit was made. Although their ideas were simplistic, such as discussing how much money was typically brought in for food, it was possible for me to introduce the basic concept of profit. This again underlines the importance of sensitive teacher intervention in order to further children's understanding. It would be possible to use more precise mathematical techniques in order to analyse the situation more accurately with older children. After the price was set, the children decided to make posters to advertise the event and ensure it's success. After a complete sell-out they were proud to find they had succeeded in making a substantial profit.

Communicate Results

The children discussed all the different aspects of the activity and although the event was a total success, the children were motivated to evaluate their achievements. They discussed, for example, how demand for the product exceeded their expectations.

The potential for such activities to contribute to mathematical development is clear. In this example alone the mathematical skills involved include Data Handling, Weight, Money, Number and the essential ability to apply mathematics to a real life situation.

4. CONCLUSION

From both examples it is clear that it is profitable to have a modelling view on children's activities, to be able to interpret these activities accordingly and to help children in their problem solving. We have become very aware of the contribution that a modelling approach can make to a child's mathematical development. It has, therefore, been possible to provide appropriate stimulation in the Early Years environment. This suggests that if all teachers could gain a similar insight as part of their training there would be positive implications for all children's mathematical development.

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18

Modelling Growth Heuristically

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SUMMARY

AIDS has been with us for more than a decade, but it has only been within the last few years that the public has recognised it as a crisis. The same can be said about the problems of our national debt and over-population in the third world. All of these problems, which seem so different in their causes and effects, are problems of growth. They started small and became large and will become larger still unless we do something to stop them. It would have been easier to solve them while they were small, but we would have had to recognise them sooner. So the question is "*How did we go from small to large-almost before we knew it?*"

If we look for an answer by drawing a graph, it would start low and end high, but then we would have to decide how to connect these points to show growth happening–almost before we know it.

We could go from small to large

quickly: {A, B, C} or slowly: $\{D\}$

We could go

suddenly: {A, C} or gradually: {B}

But there is only one way we can go–*almost before we know it*: $\{C\}$

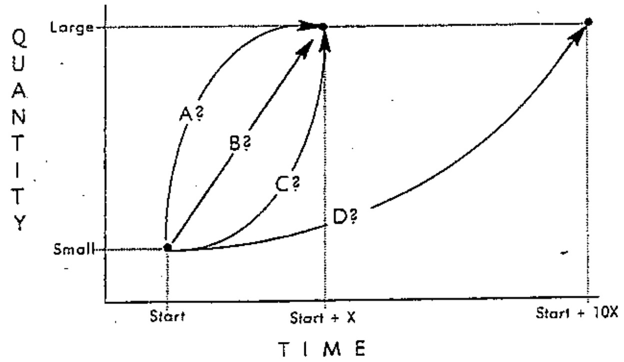


Fig. 1

Graph C shows us how the spread of AIDS, and the problems of debt and overpopulation happen because there is no warning at the start or along the way that they will become large until the end. This is what makes their early recognition and prevention so difficult and all the more necessary. But giving people warnings will not suffice to get them to make the personal sacrifices and changes in lifestyles which prevention will require. Limiting growth will be possible only when people are educated to understand how "we go from small to large" and are convinced that it will happen not by chance but according to a predictable pattern.

In this article, we will outline for mathematics teachers on the middle and secondary school levels how they might teach about exponential growth, the type we are talking of here, not through the algebraic formulas and graphs used in advanced courses, but through the process of modelling. We will create models which, like growth, are dynamic and which students can use to generate data whose graphs are exponential. Working with these models, students will be able to derive many of the basic properties of exponential functions which they will study later in advanced courses. Through these models they will come to understand several of the most important and interesting applications of exponential functions that explain the problems of growth. But most importantly, students will be convinced of the need for limiting growth to prevent the problems which result from it, because these models will show how they can happen.

1. UNDERSTANDING THE PROBLEM OF GROWTH

Let us look at the growth curve for the world's population (Fig. 2), and ask what it tells about history.

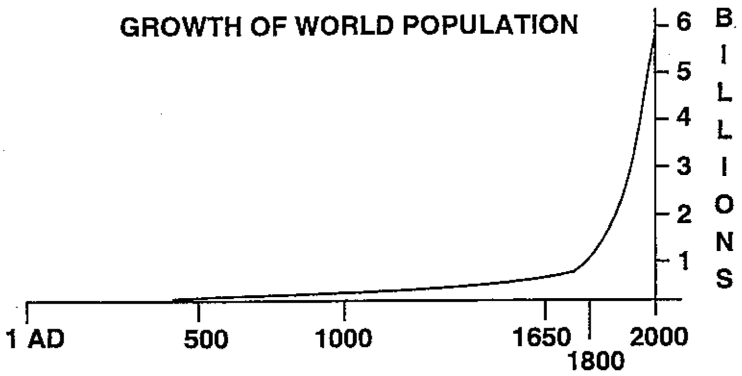


Fig. 2 Growth of World Population

One observation is that the slow steady growth of population over the centuries seems to have been unaffected by all of the wars, the famines, and the plagues of the past. But even more remarkable is the sudden upsurge in population that started around 1650 and continues today. The most significant feature, however, is what it tells us about the future, for if growth continues it will lead us into an era of scarcity which, as the steepness of the graph indicates, will happen sooner than we expect (almost before we know it).

What was the cause of this upsurge? In noticing the fact that it began around the year 1650, we might look for an explanation in historical events like the colonization of the western world, or the religious reformation, or in advances in technology and medicine which started then. There is no doubt that these events did affect population trends. However, the appearance of the graph is deceiving. If we apply the laws of mathematics, we would have to conclude that it is growth itself which is causing more growth. In other words it is just the normal process of people having children in each and every generation which is the primary cause of what appears as an abnormal upsurge in population. But how could this upsurge occur when people today are having no more children than before the year 1650?

The different ways this growth data is displayed (Fig. 3) makes the

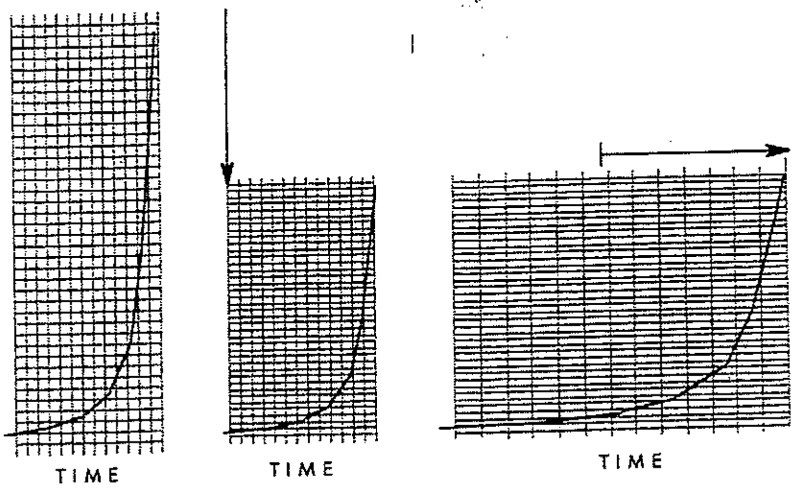


Fig. 3

upsurge appear at different times. This shows that there is no upsurge in the rate of growth, but that the rate over time is continually increasing. Therefore history, economics, or medical science alone cannot explain the growth we are observing, but that the long period of slow growth followed by a short period of rapid growth happens not by chance but according to a pattern of growth itself.

A look at some statistics (Fig. 4) on the spread of AIDS shows an upsurge in the number of cases in the late 1980's following a long period of slow growth. This is the same pattern we saw in the growth curve for population. Here again we tend to look for the cause in greater drug use, sexual contact, or in a new strain of the virus, but such evidence is lacking. People today are doing just about the same things they were doing before the upsurge as shown by the fact that the rate of transmission, the increase in the number of cases per year as a percentage of total, has not increased in the intervening years. As with the population curve, the long period of slow growth in the spread of AIDS leads to the short period of fast growth not by chance but according to a predictable pattern. If we had understood this and were convinced of it in the early 1980's, we might have been able to prevent the increases we are seeing today.

We can also see this same pattern in the growth curves for government debt. The problem which all of these examples present is not just that

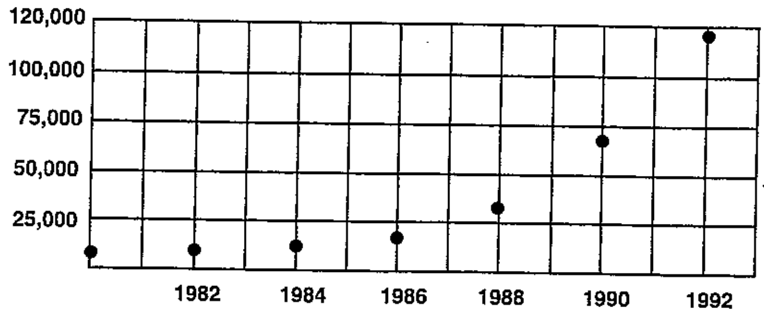


Fig. 4 The Spread of Aids

these long periods of slow growth allow people to think that slow growth will continue, it is in the way people think about growth over time. Our intuition tells us that growth in the future will be like growth in the past. More precisely, people tend to think of growth as proportional to time (Fig. 5). If a quantity grew by 2 units last year and continues to grow in the same way, then it will grow by another 2 units in the coming year. We assume that if there is a difference, then it is due to some outside factor affecting the process, but not the process itself. The growth we are observing however, is not proportional to time. It is proportional to the quantity that is growing. Here we measure growth by a percentage of the quantity at the time it is growing. So if a quantity grew by 2% over last year's level and grows another 2% again this year, we calculate this year's increase over this year's level. For example, if last year's growth was 2 units, then this year's would be 2.04 units. We are not adding 2's each year, but multiplying by 1.02 two times. The difference is slight over a short time span, but when we multiply again and again over a long span, it results in the upsurge we are seeing.

2. THE DOMINO MODEL

If we are to teach middle and high school students about growth, indeed if we are to convince them that there is a connection between the long period of slow growth and the short period of rapid growth, then we have to show them how it happens. Formulas and graphs alone cannot do this, because they are descriptive. What we need is a demonstration which like growth is dynamic, is intuitive, and which teaches students by giving them experiences they can participate in. What we need to do is model growth heuristically. So for this purpose we can show

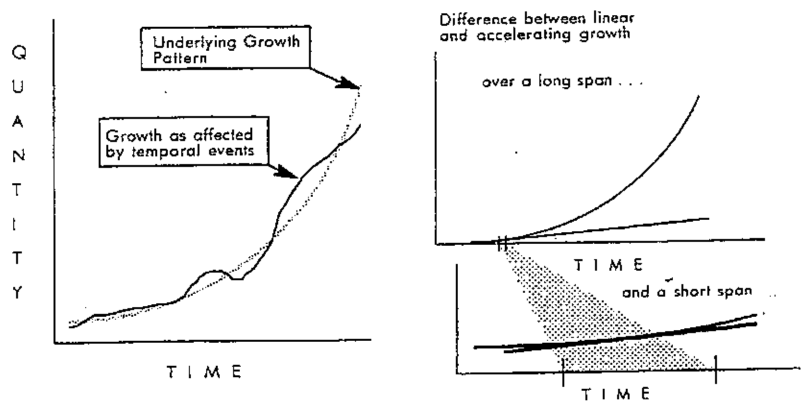


Fig. 5

students a short video, Falling Dominos and the Spread of AIDS, of an arrangement of 1,700 dominos that falls down in a chain reaction, not one by one, but in waves of increasing numbers within 5 seconds (Fig. 6). By showing how quickly one domino causes many to fall, we show by analogy how quickly AIDS can spread in a population.

This video, which was produced for broadcast as an AIDS prevention advertisement, can also be used to teach about growth in mathematics courses. But the first thing we should ask, given that it has a dramatic impact, is if it convinces students that what it shows can actually happen? We must be aware that all kinds of fantastic effects can be created by computer simulation on video and that children seeing them do not relate these to the real world. Another question is the following: Given the fact that so many dominos fall so quickly, how can we observe the pattern of a long period of slow growth followed by a short period of rapid growth, in the model that we observed in the statistics for the spread of AIDS and the growth of population and debt?

To convince students that what they saw can actually happen, they must have the experience of modelling what they saw in the video. So giving sets of fifteen dominos to teams of students, we have a competition to put them in an arrangement with equal spacing between the rows of dominos, to fall the fastest in a chain reaction. Here are some of the possible arrangements or patterns (Fig. 7). We can know the winner in a second, but after it is over, we have an opportunity to analyse how the dominos fall within each pattern. In looking at the a linear pattern A, we see that the dominos fall one by one; in the

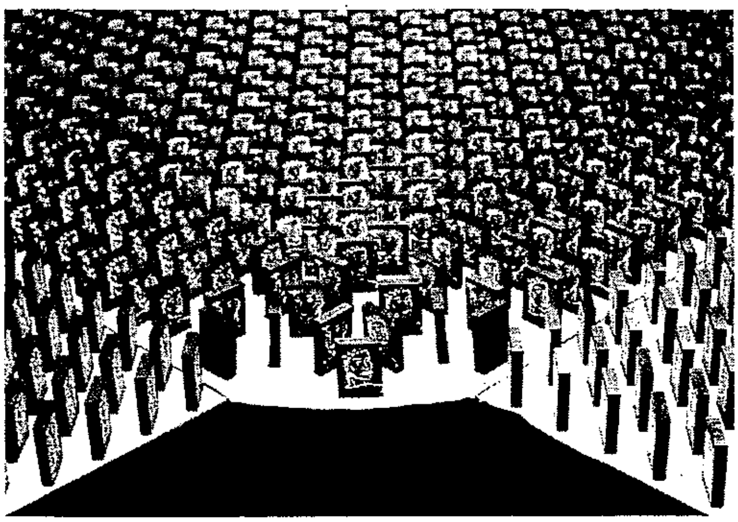


Fig. 6 From the video ***Falling Dominos and the Spread of Aids*** **©1993 John Golobe**

second pattern B which falls down more quickly, dominos are falling in groups whose numbers increase like an arithmetic series 1+2+3+4+5. In the third pattern C we have groups of dominos falling in a geometric series 1+2+4+8. If we call the groups of dominos falling simultaneously "tiers", then the total in pattern C has four tiers and the second pattern has five, so pattern C falls more quickly. Of course the linear pattern A falls the slowest. Once students have seen the video and have tried the model, they are ready to learn more about the concepts behind the models. The first is to find the relationship between time and the total number of dominos which have fallen, so we will have students record these numbers with the time on a table (Fig. 8) and make a graph (Fig. 9). When we compare the graphs, we see not only that pattern C falls in the shortest time but that it starts slowly and finishes quickly, because almost half the total falls with the last tier. We will go into more detail about this later, but here is the pattern we observed in real growth phenomena. It is important for students to gain an understanding of the structure of these models because we want to extend them to make predictions about the future course of events. So we should ask students "How many dominos would you add to each pattern to extend them to the next tier?" The first pattern is easily extended and so is the second, but the third presents a problem because when we try to add more dominos to make the fifth tier (Fig. 7), they get in each others way. We can't extend this pattern in the physical sense, but when we represent the numbers of dominos in the tiers as terms of a sum, then we can extend it numerically by adding terms inductively.

$$\text{PATTERN C: 5 TIER TOTAL} = 1 + 2 + 4 + 8 + \mathbf{16}$$

Students can also extend the other patterns inductively and record the results on a table and then graph them.

The graph itself is a model which offers us new means of analysing growth. As models, it should be possible to extend the graphs without looking at the tables of values. Here is the kind of inductive thinking that student would need to do. The numerical models were extended by increasing the last term of the sum. The graphical model will be extended by locating the next point (Fig. 10). The graph of the linear pattern is easily extended by increasing the height of every column by 1. So to locate the next point on a coordinate plot, we go over 1 and up 1. The second pattern B that is the arithmetic pattern has an increase of 2 in the second column and 3 in the third. So we locate the y-coordinate of the next point of the plot by adding a number equal to the number of the point in the sequence we are plotting.

Extending the graphical models pointwise gives us an opportunity to draw the slopes of the graph to see how the direction of the graph is changing, (Fig. 10), and with it measure how fast the totals in each pattern are increasing. But most importantly , we have an opportunity to introduce the concept of "rate of increase", as represented by the slope of a graph and use it to measure the growth of the totals over time.

We are employing a sophisticated concept of slope to represent the rate of increase without using algebra or calculus. We are not even using the slope formula. We are doing it very concretely and intuitively, and when we do it early with a model like this, we plant the seeds for later instruction. Then when we do it formally, students will understand it better. They will have an appreciation for what it means ("Oh yes, we were doing the modelling with the dominos and the slope of the graph tells us how many dominos per row were falling.")

Now lets extend the third graph (Fig. 10) and see how its slope changes. We can answer the question about locating the next point without looking at the table of values. What is happening to the rate of increase

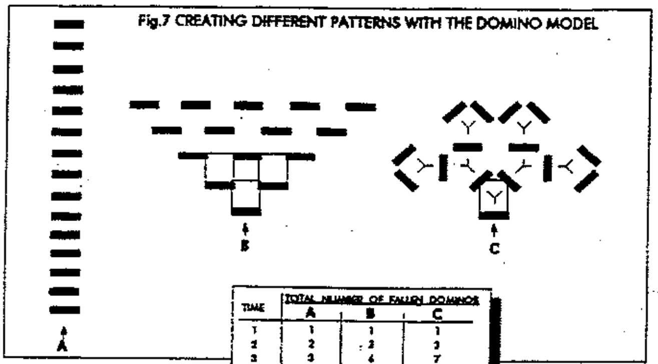


Fig.7 CREATING DIFFERENT PATTERNS WITH THE DOMINO MODEL

Fig. 8 TABULATING DATA

TIME	TOTAL NUMBER OF FALLEN DOMINOS A	B	C
1	1	1	1
2	2	2	3
3	3	6	7
4	4	10	15
5	5	15	
6	6		
7	7		
8	8		
9	9		
10	10		

Fig. 9 COMPARING THE GROWTH OF DIFFERENT PATTERNS

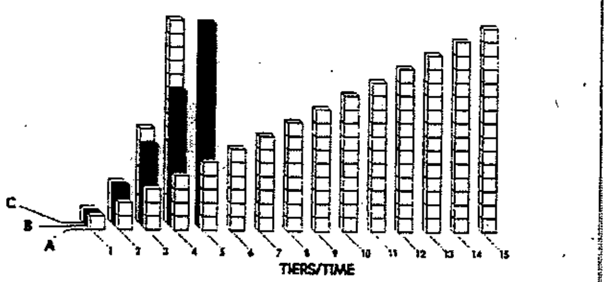


Fig. 10 COMPARING THE GROWTH IN THE GROWTH RATES FOR DIFFERENT PATTERNS

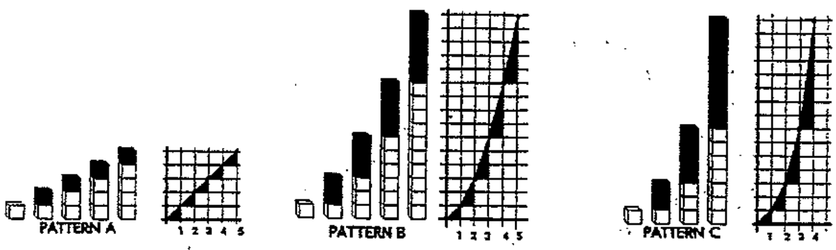


Fig. 7 - 10

in going from one term to the next? Since the third term increases by 4, we should write this increase as $\Delta_3 = 2^{3-1}$ to make it clear that we are computing the increase in the y-coordinate of the third point. So the increase in the fourth

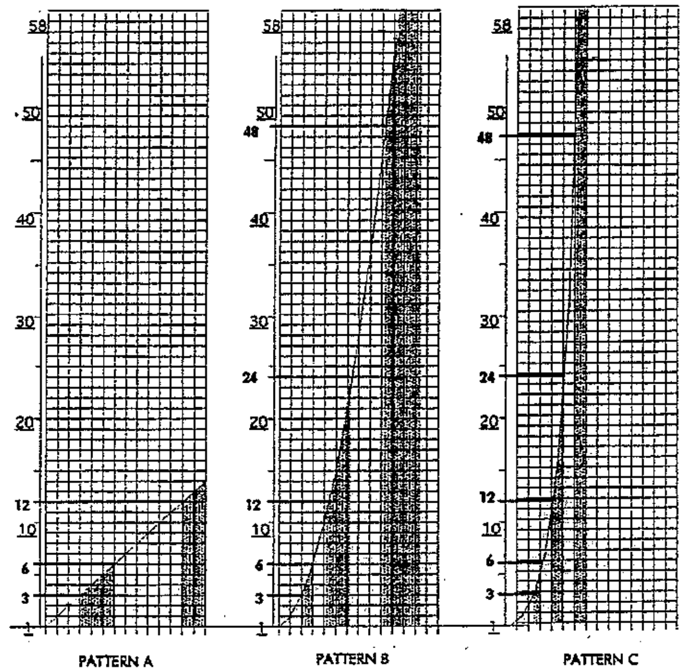


Fig. 11 Extending the Model by Induction and Comparing the Doubling Times for Different Patterns

point's y-coordinate is $\Delta_4 = 2^{4-1}$. Now we have an inductive formula for extending the graph: $\Delta_n = 2^{n-1}$.

We have introduced the concept of slope for showing in what direction the graph is going and for measuring the rate of increase in the total. But another observation we should make about the graphs of patterns B and C is that their slopes, which are the rates of increase in the totals, are themselves increasing. These are examples of accelerating growth which we usually don't see on an elementary level, and not until the second derivative is taught in calculus. This is important because we want to find out what happens to the graphs when they go higher. We can see plainly that graph of C goes higher than B by extending them,

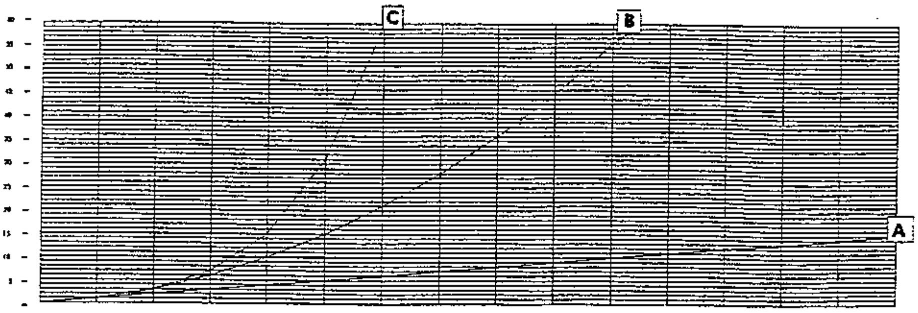


Fig. 12 Rescaling the Graphs of Different Patterns to Compare Their Slopes at the Upper End

but not that its slope is increasing faster than the slope of B (Fig. 11). This is because when the slopes go higher than 4 or 5, the increase in the angle of inclination becomes less and less. For instance a slope of 45 looks like it is parallel to a slope of 55. Only when the slopes are in a range of about 1 to 4 can we really see a difference in their angle of inclination. But it is at the upper end that the graph of pattern C pulls way out ahead of the other two (Fig. 11). So what we can do to highlight the changes in the totals when they are large is to shade in the intervals of times it takes for them to double. Starting with a total of 3, we show when it reaches 6 by shading the interval of time it takes to go from 3 to 6 and then from 6 to 12, alternating the color of vertical bands from dark to light. We see that the linear pattern A does not get very far. The second pattern B, which is quadratic, doubles four times within 13 units of time. When we compare the two graphs we see that the bands increase in width but the increase for pattern B is less, so it takes less time for its total to double than for A's. Pattern B's rate of increase (slope) is increasing faster than pattern A's which is constant.

Now lets look at pattern C (Fig. 11). There the doubling periods are narrower, so its total is doubling in less time or faster. But the important thing to notice is that these widths don't get wider and so the doubling periods are constant. Hence this type of growth's rate of increase is increasing faster than the others even though we cannot see it from the graph.

With growth that has constant doubling periods, we will always observe in rescaling its graph the characteristic pattern of a long period of slow growth followed by a short period of rapid growth. This is because with 50% of the total growth occurring within the last doubling period, the initial 50% of the growth must be spread out over all the other doubling periods which precede it. If there are 10 doubling periods, this would mean the last 50% of the growth occurs within 10% of the time and initial 50% of the growth occurs within only 5% of the time while initial 50% is spread out over 95% of the time. With 20 doubling periods, the last 87.5% of the total growth occurs within the last three doubling periods which is 15% of the total time. By carry the doubling process further, more of the total growth is squeezed within a shorter portion of the time making for the short period of rapid growth. Herein lies the danger we have talked so much about.

We have strayed in our analysis from the heuristic modelling we need to make our predictions credible, so let us return to graphs of the dominio model. There is yet another way to compare the slopes directly when these graphs have large totals. We can do it by "rescaling" their graph (Fig. 12). What we have done is compress the vertical axis and stretch the horizontal axis. This makes the slopes at the upper ends have different angles but does not change their numerical values. Lets look at an example. The 8th increase of pattern C has a slope of $\Delta_8 = 2^7$. If we compress the vertical axis by 1/4, it appears that we are dividing the numerator of the slope by 4. If we stretch the horizontal axis by 8, then it appears that we are multiplying the demoninator of the slope by 8. So these changes together make it appear that the slope of the corresponding segment on the rescaled graph has a slope of $2^7/(4)(8)$ or 4. The slope of graph C can now be distinguished from that of B by the way it bends upward.

3. FITTING THE DOMINO MODEL TO REAL DATA

These graphs are meant to represent the ways dominos fall down, and that was meant to model the way AIDS is spread in the population. It is clear which pattern falls the fastest, but the question is which model best represents the spread of AIDS? To answer this question, we will rescale each of the graphs A, B, and C to see which best fits the data on the spread of AIDS (Fig. 13).

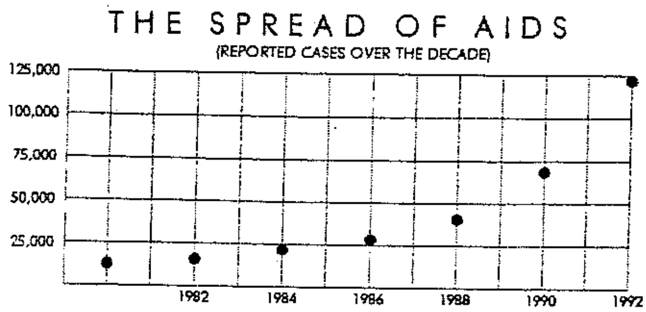


Fig. 13

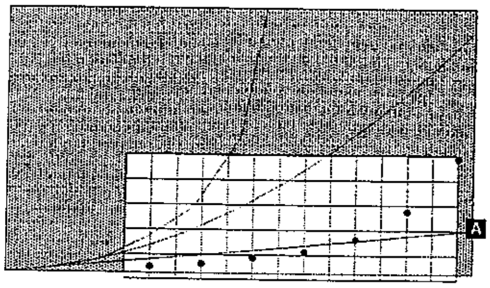


Fig. 14

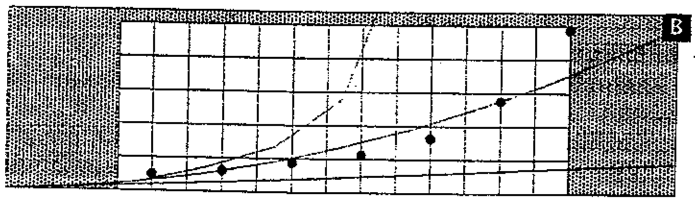


Fig. 15

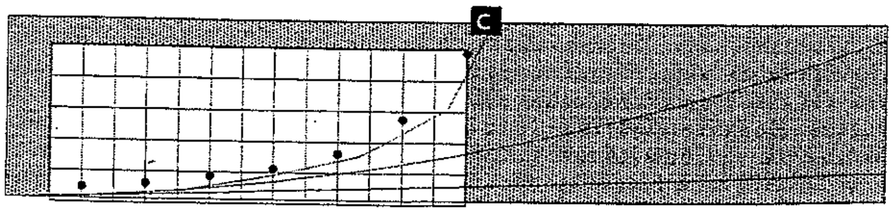


Fig. 16

When we say "fit the data", we mean have the graph go through or near as many points as possible, preferably at the high end. We should mention here that rescaling of a graph can be done dynamically on a computer that operates with Macintosh or Windows. By selecting the graph and grabbing the corner handle of the "marquee" it is in with the "mouse", and then by dragging it diagonally down to the right, all of the forms shown in Figs. 14, 15 and 16 can be obtained in one continuous motion. This represents the simultaneous compressing and stretching of the graph. The dark rectangle is what happens to the box which contains the graph. With this kind of control, we can adjust the shape of the graph to fit the data by watching where the curve goes while dragging it. We don't need to solve systems of equations. It is all very visual and intuitive. What we see here is that this linear graph goes near the first points of the data (Fig. 14) but misses the ones at the high end. So it cannot be a very close fit. We have a better fit with the second graph B (Fig. 15). It works well with more of the data points, but again misses by a wide margin at the top. And finally the third graph (Fig. 16) fits the data most closely.

So the pattern of dominos which falls like a geometric series really tells us a lot about what is happening in the spread of AIDS, at least as far as this data goes. Does it have the pattern we are looking for? Yes, it has a long period of slow growth followed by a short period of rapid growth. Is there any connection between those two periods? Yes! They are part of one function which is the point we are trying to make. Have we convinced students that the spread of AIDS puts everyone who engages in sexual activity at risk and that they should modify their behavior? This is a question for students to answer.

4. MODELLING PROPORTIONAL GROWTH

The model of growth based on the domino pattern is intended to show how a long period of slow growth leads to a short period of rapid growth. However what may appear like rapid growth on a small scale (Figs. 17, 18), may not appear to be rapid on a large scale

Comparing the Growth of Totals in Patterns

$$\mathrm{A}:\ 1+1+\ldots\{y=x\},\qquad \mathrm{B}:\ 1+2+3\ldots\{y=x^2/2+x/2\},$$

$$\mathrm{C}:\ 1+2+4+m\{y=2^x-1\}$$

COMPARING THE GROWTH OF TOTALS IN PATTERNS

A: 1+1+ ... [$y = x$], B: 1+2+3 ... [$y = x^2/2 + x/2$], C: 1+2+4+... [$y = 2^x - 1$]

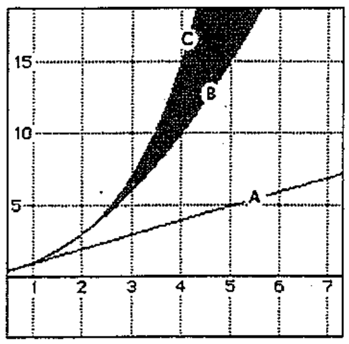


Fig. 17

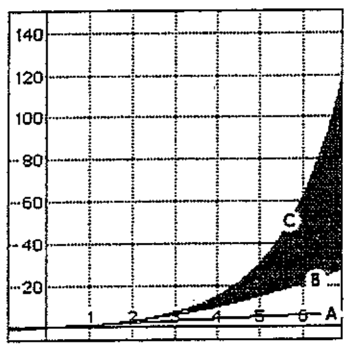


Fig. 18

The graph of pattern B, which is quadratic, looks like it is exponential on a small scale (like C), but linear (like A) on a large scale. Therefore appearance alone cannot adequately describe growth of the type which characterises the spread of AIDS and the problems of overpopulation and debt. The characteristic which distinguishes this type of growth from all others is that its rate of increase is proportional to the quantity that is increasing. Mathematicians would start with this definition and derive the exponential growth function:

$$\frac{f'(x)}{f(x)} = k \leftrightarrow f(x) = ae^{kx}$$

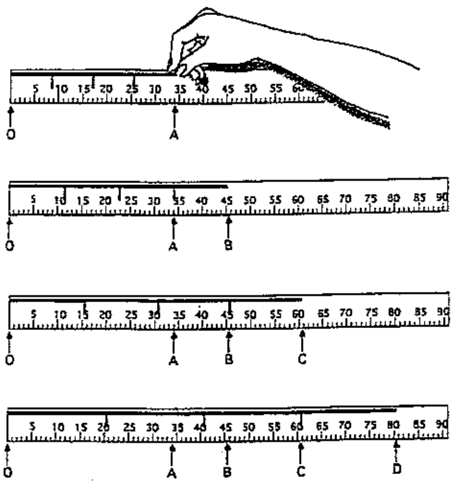
We could have derived a similar proportionality by induction. However, we would like to base our model for exponential growth on this definition, too, not for the sake of mathematical rigor, but because this is the way things grow in nature. Living organisms tend to grow in proportion to their size, as do populations of organisms. Disease tends to spread in proportion to the number infected. There are environmental factors which limit growth, but growth in nature is essentially exponential. We can find man-made relationships which are exponential, as with the compounding of debt. It is this proportionality which characterises growth - the more there are, the more there will be - that causes the upsurge which in turn causes the economic, social, and environmental problems that we are witnessing.

We could simply adopt the mathematicians algebraic models, but we also want to show students how growth in nature actually happens. So what we want is a model for proportional growth which is dynamic, which is intuitive, and which teaches students by giving them experiences they can participate in. What we want to do is model proportional growth heuristically.

Such a model can be created by attaching little markers to an elastic band at equal intervals. In the diagram below (Fig. 19) three such markers divide the the length from 0 to A into fourths. When the end of the the band at A is stretched to position B, so that the third marker coincides with A, the original length OA is now divided into thirds, so that the new length OB represents a growth of OA by one-third; OB=(4/3) OA. By repeating this process from B to C to D, the lengths increase by one-third each time. This may be represented by multiplying the original length by the improper fraction 4/3 three times: $OD = (4/3)^3(OA)$.

What students see in using the rubber-band model is direct evidence that while the increase becomes greater each time they do a stretch, it is always proportional to the length that is increasing. They can understand how proportional growth starts slowly and then accelerates, because the rubber band model, like the domino model, shows how it happens. And when exponential growth is represented algebraically and then applied in studying the problems that it poses such as overpopulation, runaway debt, and epidemics, students will be able to better understand the need for prevention.

There is another version of this stretching model for proportional growth which has a dynamic to show how the rate of growth grows and gives us the ability to derive several important properties of exponential functions. But it also has the virtues of being more easily understood and used by students on all levels to construct exponential growth than algebraic models. This is the "projection model" for proportional growth. Drawn below (Fig. 20) is the model for a growth ratio of 6:5 or a growth rate of 20%. The growth of a quantity is represented by an increase in the height of a column. Since the scheme of lines divide a column in six equal segments, growth from the fifth to the sixth lines always increases the height of the column in the ratio of 6:5. But in this model, the columns are spaced so that the height a column grows to the same height as the next column at the start of its growth. So we can construct the repeated growth of a quantity by increasing its height from the fifth to the sixth line, translating (projecting) this height to



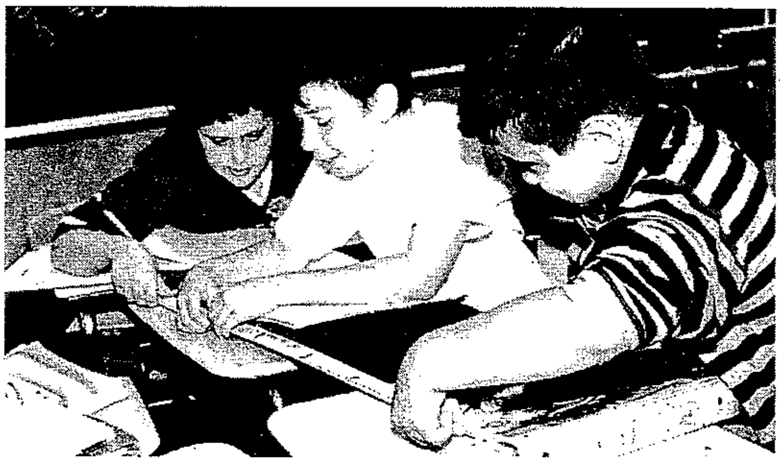


Fig. 19

where it meets the fifth line again, and then repeating this cycle. By putting all the columns together with even spacing, (Fig. 21) we have a graph of growth over time. In this graph we can observe how the rate of growth increases with time.

As a column can be positioned to meet the fifth line whatever its height, we can use this model to construct the growth of a quantity of any initial size (k). As the number of lines in the scheme of this model can be

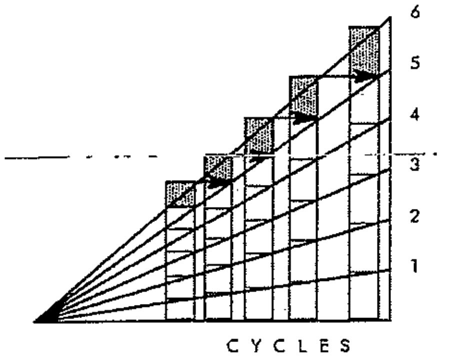


Fig. 20

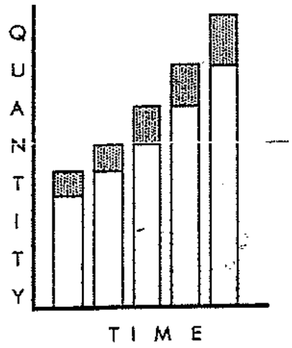


Fig. 21

varied, it can work with any growth ratio $\{a/b : a > b, a \in Z, B \in Z\}$. And as students can generate growth by recycling (n), they can construct the graph for exponential growth: $y = k(a/b)^n$. We can also plot the graph of exponential growth by calculating with this formula, but cold calculation does not give students the same feel for how fast growth increases as does the dynamic of constructing it with the proportional projection model. And when the graph is rescaled to show a long period of slow growth followed by a short period of rapid growth, then the formula for exponential growth, which is its algebraic model, will be better understood.

Several important properties for exponential growth that we might establish with the use of logarithms or later with calculus, are immediately derived from the proportional projection model. One is that all exponential growth has a constant doubling period.

$$f(x + k) = 2f(x)$$

Take one section of the model in which doubling occurs (Fig. 22). For every similar section, whether smaller or larger, doubling will also occur and it will happen within the same number of growth cycles (which could be fractional).

Because the proportional projection model is a step function, we can give students an idea of what the "integral" or area under a curve means without the veil of the "limit". For instance the area of four consecutive growth cycles (Fig. 23) could represent the cost of tuition for four years at a university with $r\%$ annual growth due to inflation.

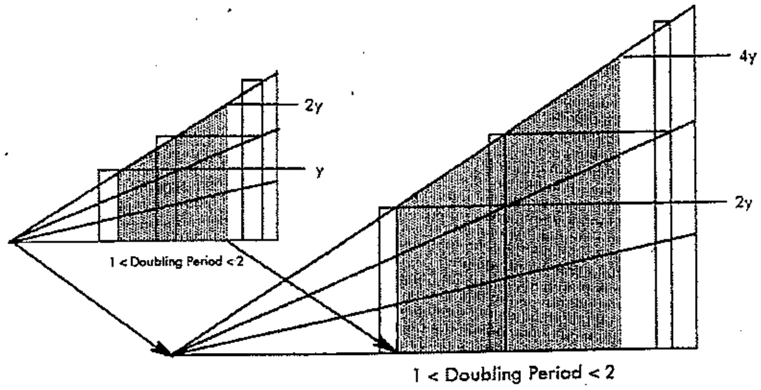


Fig. 22 The Doubling Period for $y = (3/2)^n$

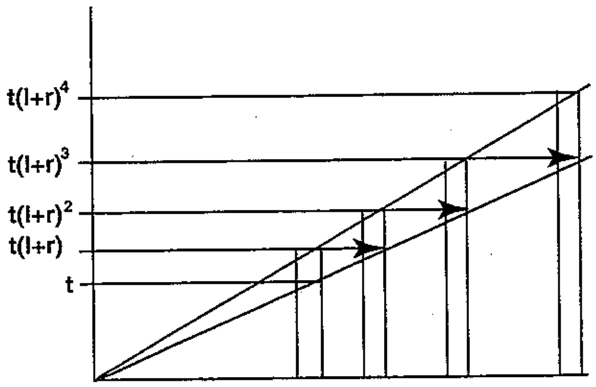


Fig. 23

By making the growth cycles more frequent, we pass to the area under a curve. This can be demonstrated with a graphing spread sheet by changing a parameter.

Because the proportional projection model works by cycling, we can introduce students to the idea of recursion before they learn about it with formal notation.

The proportional projection model is the link in the development of students understanding of growth from the concrete operation of the domino model to the abstract symbolic notation of algebra. With it we can interpret operations of exponents,

$$(a)^n(a)^m = (a)^{n+m}, (a^n)^m = a^{nm}, a^nb^n = (ab)^n$$

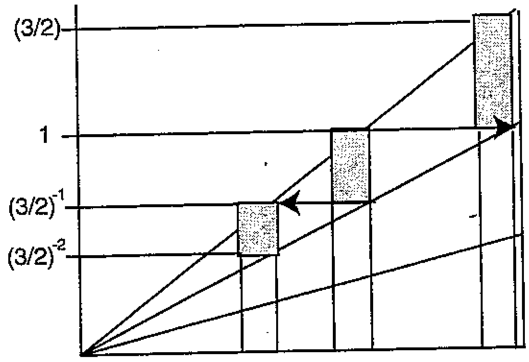


Fig. 24

as well as give meanings to the zero and negative exponents. By reversing the direction of the model, we obtain a model for decay or proportional reduction. With it we show that the inverse of 6/5 is $(6/5)^{-1} = 5/6$. (Fig. 24)

Even after exponential functions have been introduced algebraically, the projection model for proportional growth can serve as a stepping stone between the interpretation of a problem situation and the derivation of an exponential equation for its solution. For instance the problem "How much money t would you have to save at 5% interest each year if you wanted to have $8,000 in four years?",

$$(1.05)^t + (1.05)^2 t + (1.05)^3 t + (1.05)^4 t = 8,000$$

can first be represented with the four-cycle projection above by letting $r = .05$ and their total = $8,000.

A problem in exponential decay "How many coins are in circulation after four years if 1,220,000 are minted yearly and 5% are lost yearly?" can be represented by reversing the four-cycle projection (Fig. 24) and then letting $t = 1,220,000$ and $r = -.05$. With this, we can represent the total with the computation:

$$(1-.05)(1,2220,000)+(1-.05)^2(1,220,000)+\ldots+(1-.05)^4(1,220,000)$$

Being able to represent exponential growth/decay with the proportional projection model gives students the framework they need for making applications of geometric series and mean to problems with inflation, bank interest, carbon-dating, population growth and other processes which grow by recursion.

What kind of model do we need for teaching about growth? If our purpose were to have the most accurate and complete description of the growth process, we might use the model demographers, epidemiologists, and statisticians use: a logistic equation (Fig. 25).

$$f(t) = \frac{A}{1 + Be^{-Akt}}$$

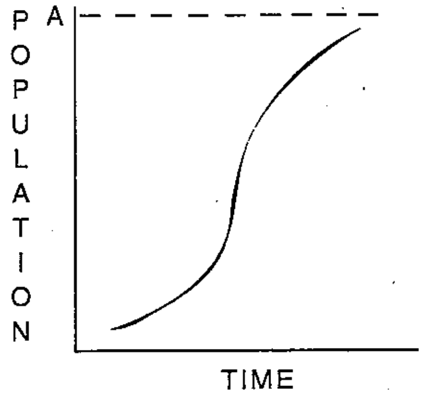


Fig. 25

Its curve can be made to fit a plot of growth data by adjusting the parameters not only up through the period of rapid increase, but also as growth slows as it reaches its limit. This model provides what the researcher needs, but our purpose is not to train researchers. It is to teach students to understand the consequences of growth before it reaches the limit. So our model has to show how this upsurge occurs and convince people that it must be prevented from happening. The domino model gives a dramatic and convincing demonstration of accelerating growth. The proportional projection models not only gives us an expanded range of growth to model, but also the ability to analyse and draw conclusions about its consequences.

19

Mathematics Projects Course in Teacher Training – Constructing Nice Puzzles

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SUMMARY

This paper provides arguments for the inclusion of project-type course into the official programme of the pre-service training of mathematics teachers. These arguments are given in the form of 6 Theses and are illustrated by using Newspaper Puzzles as modules for the course. A course schedule is suggested and main features are described. The task of assessing and evaluating is discussed.

1. INTRODUCTION

The most important factor in teaching any non-traditional subject is the teachers' attitude and educational background. This applies to the teaching of modelling and applications, as well. During the pre-service teacher training, however, prospective teachers hardly have casual occasions which would provide experience for their future profession.

Therefore, we recommend the inclusion of a "Projects in Mathematics Education" course into their educational programme, where, preferably in small groups, they could exercise the designing, developing phases of the construction of feasible school projects. Such a course could also provide a pattern on how to implement and guide application-type work in the school environment.

Acknowledging this need, and observing that the present school-practices do not favour teaching genuine applications of mathematics to real world problems, such a course had been introduced into the compulsory part of the teacher training in the Middle East Technical University, Ankara, Turkey in 1990. Our experiences with this course are embedded into the present work. As for details of the educational system in Turkey and the place of the teacher training in it, the reader is referred to Aksu (1990), Aydin (1989) and Ersoy (1992).

In this paper we are mainly concerned with the applications side of the dichotomy of "Modelling and Applications". Even here, our aim is to show how to prepare student-teachers to insert discussions on applications, items dealing with applications from their environment into the traditional mathematical curriculum, using them as illustrations or for introduction of classical mathematical questions.

We illustrate our ideas through a common "Newspaper Puzzle" named by the students as "Pascal Puzzle". Only as a side remark, we would like to mention that such puzzles were really nice brain-teasers even a decade ago, while now they need a few minutes of PC programming, only. An aspect, which should be taken seriously when planning curriculum changes.

2. PROJECTS IN MATHEMATICS TEACHER EDUCATION: ARGUMENTATION

There is a lot of discussion about the professional qualifications of mathematics teachers and their training to develop these qualifications. It is more or less generally accepted, that the professional knowledge of teachers include at least three categories:

- Knowledge of the subject, i.e. mathematics.
- Knowledge of pedagogy, in general and didactics of mathematics, in particular.
- Knowledge of management within the school environment.

The importance of these categories has different weights in views of different researchers, see e.g. Romberg (1988), Ernest (1989). Good subject matter preparation is necessary but insufficient.

Thesis 1: Sound theoretical background is only one of the prerequisities to effective teaching in any area. Without supplemented methodological, didactical knowledge, no success can be expected.

Supporting (negative) evidence is given by the examples of the teaching probability, statistics, and computers throughout the world. The present generation of teachers may have attended short courses on these areas, but these courses had no time to discuss (or even touch upon) applications or didactical questions. As a result, teachers fail to see their importance, and are afraid to include them in their classroom practise. A project course could help to overcome this difficulty to some extent: prospective teachers could deal with applications taken from their own environment, and discuss how to attack these problems in the classroom.

Our second thesis is strongly connected to this.

Thesis 2: In any established or foreseen area of education, school subject, the instruction of teaching methods should be included in the pre-service education programmes.

Many people share the opinion, that the profession of mathematics teachers requires a synthesis of mathematics and educational knowledge. As an example, McNamara (1991) claims, in connection to the reforms of teacher training in Britain and the USA, that prospective teachers must be able to convert knowledge of the subject into a teachable subject for a wide range of pupils. This view is vital regarding applications. It does not suffice to know what is applicable, but also HOW TO COMMUNICATE this knowledge to the students. A recent case study by Y. Toluk (1994) shows the practising secondary school teachers also share this view.

Again, we argue, that a good way to prepare the students for this job is a project type work, where they can discover the "Know-how" for themselves, and what is at least so important, observe the "dead-end streets" on the way to communicate the knowledge to others. This implies another Thesis:

Thesis 3: Activity, discovery approach is needed already during the pre-service teacher training in order to exemplify and secure ways to teach understanding the tasks, utility, and the function of mathematical modelling and applications.

In many countries, this is in conflict with existing teacher training programmes. A "small step" policy, as suggested by Black at al (1993), may help to introduce changes. Such a small step could be the introduction of a project course into the programme.

In our observation, projects are highly motivational to most students because of their active participation and the close contact with supervisors Students appreciate topics which orientate them not only toward the content of current school mathematics, but also relevant to their future job. Our next thesis comments on this.

Thesis 4: In the classroom, examples of applicability must have actuality. Future teachers should be able to select appropriate problems from their "present time" surroundings. There is a need for "ever-green", always actual tasks.

An extensive bibliography on applications and modelling topics, which can be successfully treated at school level, is given by Kaiser-Messmer *et al.* (1992). The importance of ever-green problems lies in the fact that the treatment during the pre-service period can be copied during the in-service years. Pupils will view such problems as actual ones, and the preparational working time of the teachers, especially during their first practising years, is considerably decreased. Our "Module" example Pascal Puzzle is an example for such ever-green tasks. A rich source of such tasks can be found in the area of secret codes (see Nemetz, 1991).

Thesis 5: Teachers' search for real world problem should be governed by the following main steps:

a) Fix the conclusions to be arrived at.
b) Collect matching problems.
c) Analyse prospective models and methods of solution and choose one of the problems for the classroom.
d) Sequence the tasks and organise classroom discussions.
e) Be prepared to direct the process of making conclusions and formulating the results.

If teachers do not get accustomed to this process during their training period, they will not have time and will not be able close up later. It may be interesting to note, that a different way was more natural to course students. They followed a similarity principle: The Newspaper puzzles of Example 2 and 3 were collected by students.

Work in the project course should proceed along the lines that could be followed in the school practise. Our view in this respect is formulated as

Thesis 6: Tasks for school pupils should be organised along 3 main steps:

a) Understand the important features of the problem.
b) Discuss, analyse the problem, discover applicable methods.
c) Formulate the conclusions.

Here the last step usually does not get the necessary attention. It cannot be stressed enough how important it is to have an easy-to-read, narrative formulation.

3. PROPOSED COURSE SCHEDULE AND MAIN FEATURES

We have found it useful to organize the project course for student teachers into three periods.

Period 1 (2-4 weeks): By frontal instruction, discuss the general goals of the course and offer a variety of application tasks while specifying the immediate educational goals. Let the students choose one of them according to their interest. By further frontal discussion, provide the necessary theoretical background in a concise form. Prepare a working plan for the second period.

Period 2 (4-6 weeks): Independent or small group work: Library search, computer use, consultations, joint discussions.

Period 3 (3-4 weeks): Writing-up, reporting. Oral presentation by the students.

This time grouping is likely of general validity. Specific to teacher training is the emphasis on the exemplifying of views about designing and developing materials for school use. They are as follow:

- Encourage active participation of students instead of letting them to stand by in an observational capacity.

- Activities should be designed to provide stimulation and challenge to students.
- The course should be based on solving school level real problems by scientific enquiry and other intellectual activities which require critical thinking from the students.
- Widely available technology should be used during the entire work to pave the ways for school utilization.
- Student teachers should feel the necessity of designing and constructing worksheets and handouts as teaching aids.
- Modules treated should connect different topics within mathematics and should form interdisciplinary links with other school subjects.

4. CRITERIA FOR ASSESSMENT AND EVALUATION

The essence of assessment in the project course is its continuous nature. This continuous assessment focuses on the following points and activities:

- Response to weekly scheduled sets of questions and tasks by the entire group or its members.
- Evidence of study of the mathematical topics related to the particular project.
- Developing teaching aids and materials.
- Finding possible project topics, collecting samples of materials.
- Written documentation of the work and the findings.
- Oral presentation of the report in front of fellow students.

The written documentation is prepared in two steps:

- an initial submission by the group or the individual, consisting of a statement of the problem under investigation and an outline of the working method (in about one page), and
- a final report, giving details of the mathematical analysis and educational aspects of the project topic, and the applicable teaching and learning materials.

Note: the crucial issue in such a course is the problem of giving credits to the students and assessing them. The authors would be happy to collect suggestions and share experiences.

5. ILLUSTRATING EXAMPLES

Our ideas are illustrated here by a common "Newspaper Puzzle" named by the students as "Pascal Puzzle". This was originally intended to discuss linear system of equations. It involves arithmetics of digits, with a few unknown digits, only. Such a puzzle is called NICE if it has a unique, non-negative solution. The simple problem of solving the puzzle evolved to become more complex: the task was to construct NICE puzzles. The graphs we are using may be considered ugly ones. We use them, however, purposefully: the most simple text editor can be used to reproduce them, they do not suppose any knowledge of computer graphics or programming.

Example 2 and 3 were collected by students, and the possible educational goals were derived after the examples had been collected. We simply edited their suggestions without adding any additional comments.

EXAMPLE 1: PASCAL PUZZLE

This is a common newspaper puzzle offered regularly for children under different names. This regularity makes it "ever-green". A triangle shaped figure is given with empty boxes, except a few where usually positive numbers are given. The task is to fill in the empty boxes with numbers in a way, that every number is the sum of the numbers immediately under it. Instead of the customary fancy forms, we illustrate this problem by a figure which is easy to implement by a text editor, i.e. in a way, which is accessible to students and teachers.

```
                 |_______|
             |_______|___7___|

       |___4___|_______|_______|
   |_______|___2___|___3___|_______|
```

The solution is straightforward:

```
       |___|                    |___|                    |_ 16 _|
     |___|_7_|               |___|_7_|               |_ 9 _|_ 7 _|
   |_4_|_5_|___|           |_4_|_5_|___|           |_ 4 _|_ 5 _|_ 2 _|
 |___|_2_|_3_|___|       |_2_|_2_|_3_|___|       |_ 2 _|_ 2 _|_ 3 _|_ -1 _|
```

Of course, we do not wish to suggest the solution of such a problem as a task of the project course. This problem can be utilized in a different way. The construction of such puzzles attracts the attention of students, and it can direct future teachers to generate a number of similar individual problems for classroom use. In this respect we have to quote our students who have found the above example to be "not a nice problem", because one of the boxes in the completed figure contains a negative number. They have defined a nice puzzle as one which contains positive digits in the initial setting, and which has only positive entries in the completed form. Now we can pose the task to construct nice puzzles.

A variety of educational goals may be specified, like

- exercising the solution of systems of linear equations
- finding relations between the number of unknowns and that of equations
- introducing linear independence
- pointing out the utility of the inverse matrix for solving equations with the same matrix but with a different constant vector
- writing computer programmes to solve linear equations
- comparing different methods for solving systems of linear equations
- discussing the complexity of computer programmes
- finding all "independent" 4-places, as a combinatorial problem
- designing algorithms to generate all or a randomly selected independent 4-place
- designing algorithms to fill in all possible ways a chosen independent 4-place, or to fill them in randomly
- writing a programme documentation
- generalize the given problem to larger arrangement
- introduce the idea of the Pascal triangle.

EXAMPLE 2: INSERT-TYPE CROSS-WORDS PUZZLES

These types of cross-word puzzles ask for completing a usual arrangement of boxes from a given list of strings of letters or digits, like the following figure and list:

	XXXX			
			XXXX	
		XXXX		

5-letter words: 12345, 13579, 52846, 92226

3-letter words: 152, 222, 314, 524

2-letter words: 42, 44, 45, 72

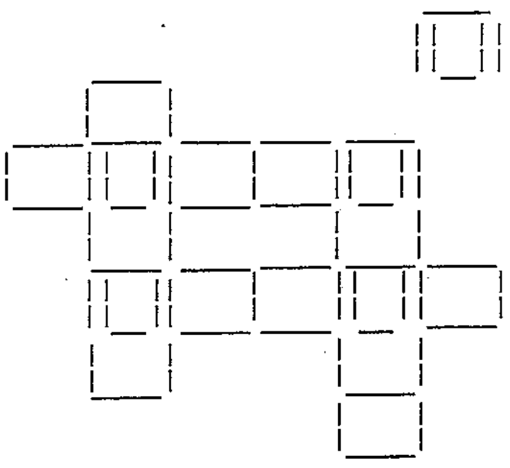
Now a "nice puzzle" is defined as one which has a single solution, with the possible exception for symmetry.

Students' activity can be directed towards

- to write down how THEY have solved it
- discuss/collect ideas to solve
- extract features for solving
- prepare algorithms to solve
- discuss how to simplify and complicate such puzzles
- prepare algorithms for constructing nice puzzles
- by simulation, find out the chance that a randomly filled arrangement provides a nice puzzle

EXAMPLE 3: MAXIMIZING JOINTS-VALUES

These are less frequent but nevertheless typical puzzles. They consist of horizontal and vertical box-bars. Their intersecting boxes are called joints, which play an important role. Again, there is a list of strings, but this time the list contains essentially more strings than necessary to fill in the boxes. Symbols in the strings have numerical values assigned to them. The task is to find a solution for which the symbols placed into the "joints" of the figure sum up to a maximum. A simple illustration is given below. Here the joints are represented by the boxes



Symbol values:

Digits:	**0 point**
Vowels:	**2 points**
Consonants:	**3 points**

Strings:

ABA66	**STVA6**
ABCD5	**WTXA8**
AX3R4	**M5678**
DB5EM	**MMNNO**
DT5EM	**01210**

The educational goal of treating such puzzles is to prepare the way for mathematical (economical) programming.

CLOSING REMARKS

In this paper we have argued for the inclusion of a "Projects in Mathematics Education" course into the official programme of the pre-service teacher training. Based on our experience, we propose that this course be run preferably in small groups, such that student teachers could effectively exercise the designing, developing phases of the construction of feasible school projects. Such a course could also provide a pattern on how to implement and guide modelling and application type work in the school environment.

Our experience is definitely positive. Nevertheless, we have encountered one problem, namely the problem of giving credits to the students and assessing them. The authors would be happy to collect suggestions and share experiences.

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20

Using the Laboratory Interface in the Mathematics Classroom-What, Why and How

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SUMMARY

Laboratory interfaces--devices that enable students to collect, store, and analyse scientific data through the use of computers and electronic sensors--are a relatively new technological development, although their use is already widespread in secondary science classrooms. Due to the recommendations of the *Curriculum and Evaluation Standards for School Mathematics* (National Council of the Teachers of Mathematics, 1989) and the resulting interests in real-world data and mathematical modelling, however, the use of interfaces is growing among mathematics educators, as well. In this chapter, we explore the use of laboratory interfaces in the mathematics classroom by addressing the following questions:

- What is the laboratory interface?
- Why use the laboratory interface?
- How is the interface used in the classroom?

1. INTRODUCTION

Perhaps nothing has influenced American mathematics education more in recent years than the development of technology. Throughout the nation, secondary students are using calculators and symbolic manipulators to complete complex arithmetic and algebraic calculations, graphing packages to visualize mathematical expressions and explore geometric relationships, spreadsheets and data analysis packages to perform statistical functions, and CD-ROM technology to generate and explore data sets that would otherwise be inaccessible. In the United States (and around the world), technology has been embraced by teachers and students alike.

Laboratory interfaces–devices that enable students to collect, store, and analyse scientific data through the use of computers and electronic sensors–are a relatively new development, although their use is already widespread in science. Due in part to the recommendations of the *Curriculum and Evaluation Standards for School Mathematics* (National Council of the Teachers of Mathematics, 1989), however, interest in laboratory interfaces is growing among mathematics educators, as well. In this chapter, we present an overview of these devices, discuss the reasons that they are growing in popularity, and represent several activities that illustrate their use in the mathematics classroom.

2. WHAT IS THE LABORATORY INTERFACE?

In science, a key to the construction of valid models is the collection of accurate data. Much of the information that we would like to gather, however, is not directly observable or quantifiable. For instance, although we can tell if something is very hot or cold by feeling it, we are unable to quantify temperature with any degree of accuracy. Similarly, although the long-term effects of radioactivity can be observed, radioactivity and its short-term effects can neither be observed nor quantified. In order to solve the problem of collecting scientific data instruments have been developed that transfer information about the physical, biological, or chemical quantity of interest to an observer (refer to Fig. 1).

All laboratory instruments that accomplish the task of measurement share the same structural design. The instrument interacts with the quantity of interest and this interaction alters the level of some indicator.

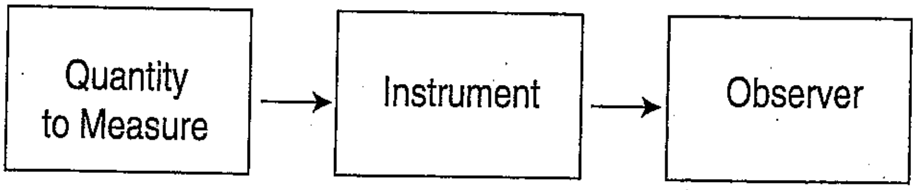

Fig. 1 The Role of Instruments in Data Collection

As an example, the horizontal or vertical level of mercury in a thermometer is an indication of temperature, whereas the markings on the thermometer allow observers to quantify this level. Similarly, the color of litmus paper is an indication of the pH of some substance, whereas charts of pH-color equivalences allow observers to assign a pH-level to that color.

Although "rough" estimates of pH and temperature can be obtained using litmus paper and thermometers, respectively, the new emphasis on experimentation, real-world data, and modelling in the mathematics classroom requires greater measurement accuracy. In order to collect research-grade data, scientists typically use electronic instruments, such as thermistors (for temperature) or pH electrodes. In general, the structure of these devices resembles that of traditional instruments.

As is illustrated in Fig. 2, electronic detectors interact with the quantity of interest to produce a small electric signal (the indicator). This signal is conditioned by simple amplifier circuits and transmitted to data processing circuits, which use an appropriate algorithm to transform the signal to a meaningful number. Read-out devices, such as meters, digital displays, or printouts, then present this number to the observer.

Although the use of electronic instruments increases the accuracy of data collection, these devices are not without their drawbacks. For one, electronic instruments are generally designed to perform a specific function. To measure pH, pH electrodes generate weak electrical signals, which are then passed through data processing circuitry to produce a pH reading on the display. Because the data processing circuitry is measurement specific, pH electrodes are only able to produce pH readings. To measure temperature, light, pressure, radioactivity, or any other attribute, other instruments are needed. As a result, the cost of equipping each laboratory station with the electronic instrumentation needed to collect research-grade data has, until recently, been prohibitive.

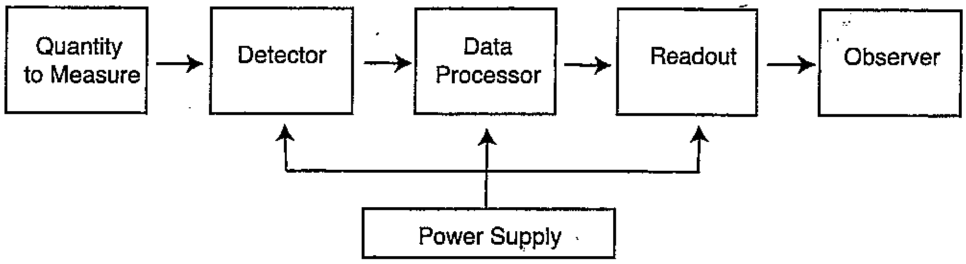

Fig. 2 The Structure of Electronic Instruments

However, the development of laboratory interface devices and the widespread availability of personal computers are alleviating this situation. As is illustrated in Fig. 3, laboratory interfaces receive and digitize the weak electronic signals produced by the sensor component of electronic instruments, enabling the computer to interpret the output of the sensor. Software loaded into the computer performs the necessary algorithms to convert these signals into readings of pH, temperature, voltage, and so on. Lastly, the computer's monitor displays the readings to observers. Thus, an interface uses only the low-cost sensor component of a measurement-specific device. The data processing and display functions of the device are performed by the computer.

In the American market, several interfaces are currently available. Two of the most widely-used interfaces are produced by SCI Technologies (1716 W. Main, Bozeman, MT 59715) and Vernier Software (2920 S.W. 89th St., Portland, OR 97225), respectively. The SCI interface is IBM-compatible, whereas the Vernier product can be used with Macintosh, IBM, or Apple II computers. In addition, Vernier is releasing a portable interface that is compatible with the Texas Instruments family of graphing calculators and is designed for use "in the field."

Complete interface systems, which include the interface, the necessary computer software and hardware, and a variety of sensors, can be purchased from either company for about $750 US.

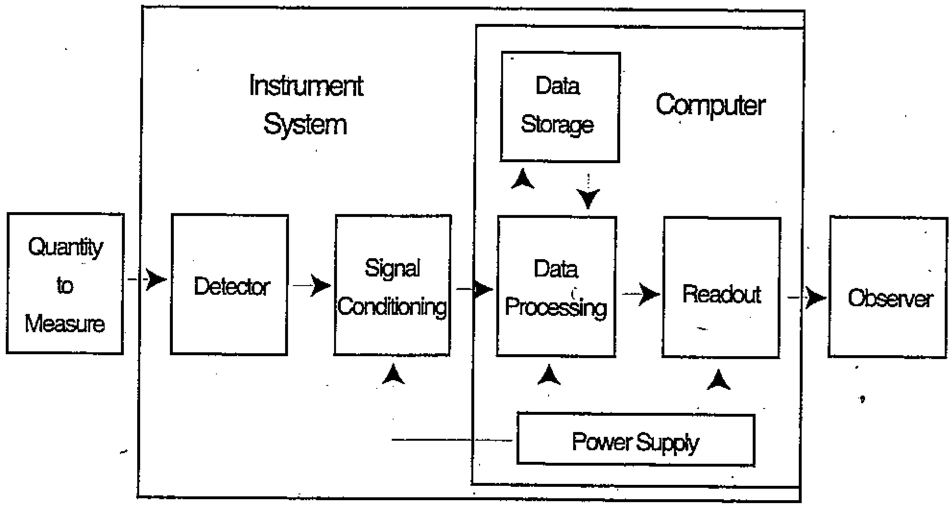

Fig. 3 The Structure of Computer Interface Systems

3. WHY USE THE LABORATORY INTERFACE?

A major advantage of laboratory interface systems is cost. Although interface systems have only three primary components (the interface, the computer, and sensors), they enable students to collect a wide variety of biological, chemical, and physical data. By comparison, comparable measurement-specific devices can cost as much as $5,000 US. Furthermore, many classrooms are already equipped with computers which makes the use of laboratory interfaces even more economical.

A second advantage of interface systems is the resultant increase in power and flexibility in the lab (Furstenau, 1991). Because data collection is controlled by the computer, students are capable of collecting data very rapidly (at the rate of several hundred per second) or very slowly (at the rate of a few samples per hour). In fact, the computer even allows experiments to be monitored overnight. In traditional classroom settings, whomever, these options are not available to students. Furthermore, the sensors that accompany most interface systems allow students to collect data that would normally be inaccessible. For example, Vernier Software offers low-cost probes that measure motion, force, radiation, light, magnetism, and pressure.

Once again, most science classrooms (and practically all mathematics classrooms) are not equipped to collect such a variety of data.

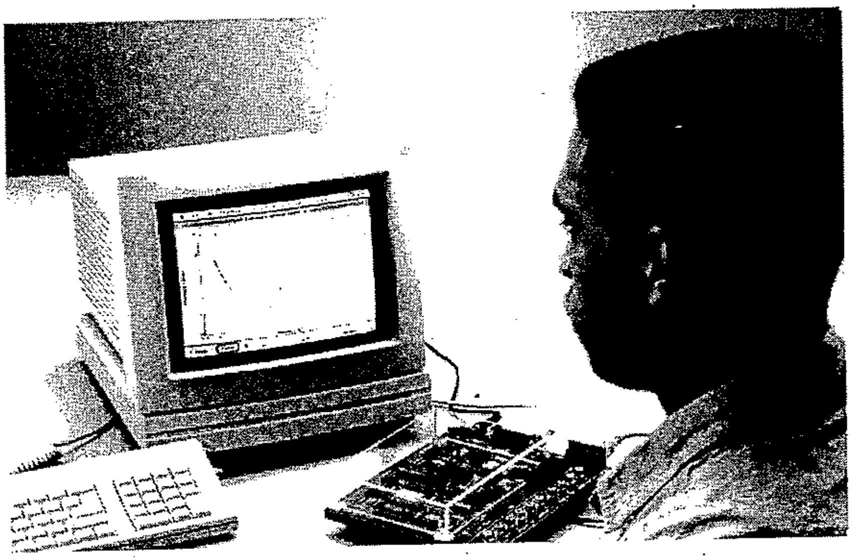

Fig. 4a

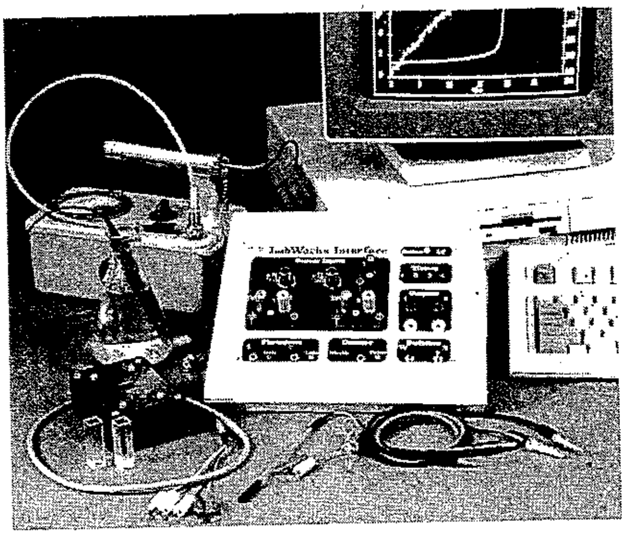

Fig. 4b

In summary, as tools to collect data, laboratory interfaces represent state-of-the-art technology. However, in the schools, experimentation, data collection, and the construction of mathematical models typically occur in the science classroom. Thus, one can ask why the use of these devices is growing among mathematics educators. Of what use are interfaces in the mathematics classroom?

In part, the rising popularity of laboratory interfaces is tied to the current reform movement in American mathematics education. The recommendations of the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) have had a tremendous impact on the objectives of American mathematics education. For instance, the Standards have refocused attention on the applications of mathematics. No longer is mathematics to be presented as an isolated set of skills and procedures. Students in the mathematics classroom are to use mathematics to explore "real" problems that arise in other subject areas.

It is this attempt to connect mathematics to other subject areas that leads to the use of interfaces. Science provides some of the richest applications of mathematics. In fact, mathematics plays such a prominent role in the development of science that many are calling for the integration of the two subject areas (Austin, Converse, Sass, & Tomlins, 1992; House, 1986; Milson & Ball, 1986). Of course, data collection is an integral part of science. To explore and understand the "problems" of science, mathematics students conducting experiments, collecting data, and constructing models. Thus, current efforts to develop mathematics within the context of science-based activities create the need to collect data. This, in turn, leads to the use of laboratory interfaces.

A related factor is the current interest in mathematical modelling. At the recent ICTMA-6 conference, several models of the modelling process were presented (e.g., refer to Fig. 5). Although each of these models characterises the process in a unique way, each reinforces the fact that real-world modelling is driven by the desire to answer questions about real problems.

In the classroom, the point at which students enter the modelling process is varied. In some settings, teachers introduce the real-world situation, pose questions about the situation, identify variables that affect the situation, and even present data that reveals the behavior of these variables. In these cases, students seek to construct a mathematical model that mimics the observed behavior of the experimental variables, use the model to generate predictions about the original situation, and compare their predictions with the actual behavior of the situation.

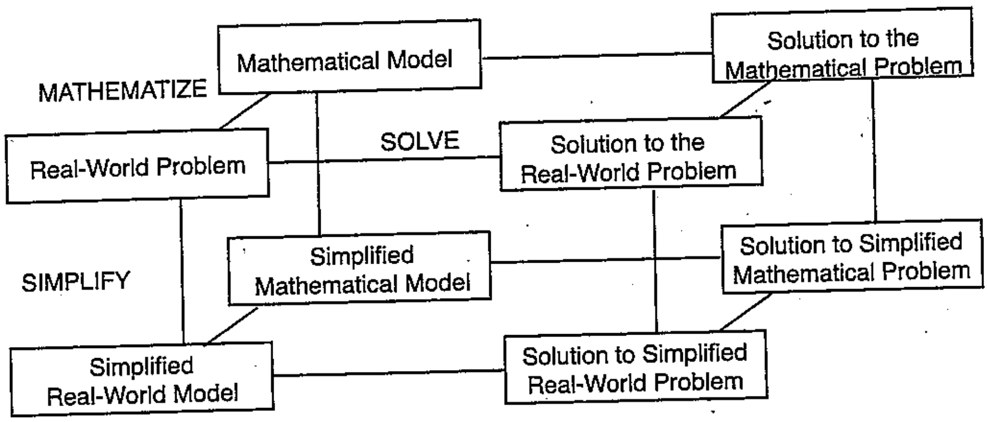

Fig. 5 Hirstein's (1991) Model of the Modelling Process (used by permission)

On the other hand, a more "radical" approach to modelling is advocated by the authors of the Standards, as well as by contemporary mathematics educators such as Gurtner, Leòn, Nuñez , and Vitale (1993). According to these authors, mathematical modelling should increase students' understanding and appreciation of mathematics, and prepare them to solve real-world problems. To accomplish these objectives, students are to be involved in the entire modelling process. That is, students identify real-world situations of interest, pose questions about these situations, identify relevant variables, design experiments that reveal the behavior of these variables, collect data, and so on.

Although both of these approaches allow students to use mathematics to solve real-world problems, the latter approach attempts to do so within a context that is "real" to the students. In particular, Charles and Lester (1982) define a problem to be a task for which (1) the person confronting it wants or needs to find a solution, (2) the person has no readily available procedure for finding the solution, and (3) the person must make an attempt to find a solution. If classroom modelling is based on contrived situations that students have no stake in selecting or desire to investigate, then we are not presenting them with real problems nor are we truly engaging them in mathematical modelling. To prepare students to use modelling to solve real-world problems, the

Standards and other sources advocate the use of problems that are real to the students.

As mathematics educators adopt this latter approach to modelling and students assume responsibility for selecting classroom activities, the role of interfaces is increased. In the absence of adequate instrumentation, students can only investigate problems or pose questions about situations for which accurate data can be collected using their own senses or traditional instruments. As a result, limitations in students' ability to collect data significantly affects their involvement in the scientific process. With laboratory interfaces, however, data collection is driven by the questions that are asked. Students are able to investigate problems that might otherwise be inaccessible and to pose questions that are real to them. This is an additional reason that the use of interface systems is on the rise in mathematics classrooms.

4. HOW IS THE INTERFACE USED IN THE CLASSROOM?

Quite simply, interfaces extend the range of situations that students can investigate. For instance, most secondary students are introduced to the mathematics of falling objects. An interesting extension of the study of falling objects, and one that leads naturally into a discussion about polynomial equations, is the study of motion down an incline plane. Specifically, questions can be asked regarding the impact that the angle of the incline plane has upon the velocity (and position) of the object.

In general, the height of an object at any time t is given by the equation $H(t) = -1/2gt^2+v_ot+h_o$, where g represents the acceleration of gravity, v_o represents the initial velocity of the object, and h_o represents the object's initial height. In the absence of an initial velocity, the height is given by the equation $H(t) = -1/2gt^2 + h_o$. If an object is rolled down an incline plane in which the angle of elevation is T (refer to Fig. 6), then the downward force of the object would be altered by a factor of $\sin T$. Thus, if the objects starts from rest, the height of the object at any time t is given by the equation $H_T(t) = -1/2(\sin T)gt^2 + h_o$. If the angle of elevation is varied, then the resulting equations represent a family of parabolic curves, as is shown in Fig. 7.

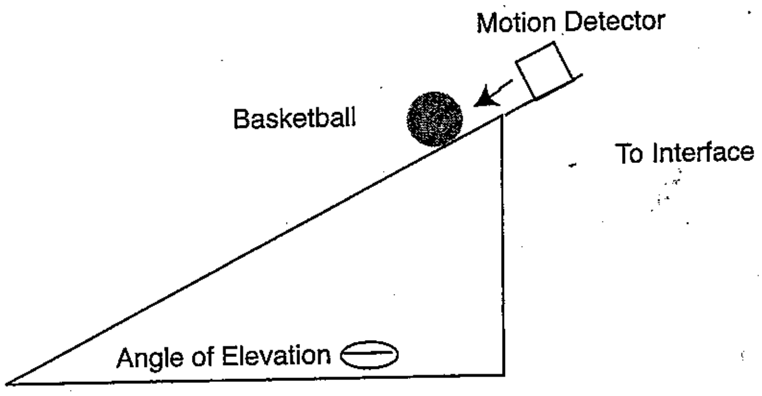

Fig. 6 An Experiment to Investigate the Mathematics of Incline Plane Motion

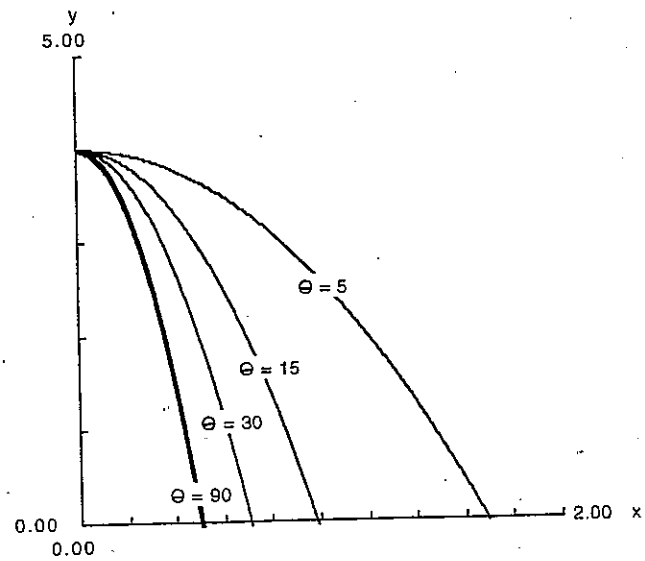

Fig. 7 Graphs of the Equation

$h(t) = -\frac{1}{2}g(\sin\theta)t^2 + 4$ **for** $\theta = 5, 15, 30,$ **and** 90

The Vernier Software Ultrasonic Motion Detector (available from Vernier for $95 US) allows students to test these conjectures and to develop the mathematics of incline plane motion through experimentation. In short, the Vernier product emits ultrasonic pulses and the time it takes for the reflected pulses to return is used to calculate the distance, velocity, and acceleration of a moving object.

In order to conduct the incline plane experiment, place a basketball at the top of an incline plane and position the interface at the top of the plane, as is shown in Fig. 6. Release the ball and allow the instrument to record the distance between it and the interface. The height of the ball at any time t is found by multiplying the sine of the angle of elevation of the plane by the distance between the ball and the interface. For comparison purposes, data collected by the Vernier Motion Detector for an angle of 15° is presented in Fig. 8. Note that the experimental curve approximates the theoretical curve, although the influence of friction results in slightly slower motion in the real-world experiment.

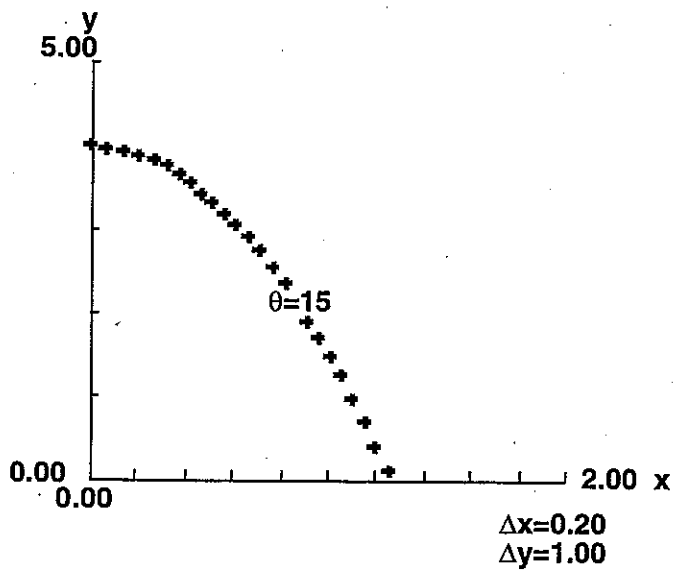

Fig. 8 A graph of incline data collected by the Vernier interface for $\theta = 15$

The study of astronomy offers an additional use of the interface, as well as an opportunity to introduce inverse relationships. Astronomy is of interest to many students, and one of the fundamental questions in astronomy concerns the relative position of objects in space. Specifically, one might ask how far a given star is from earth. Although there are several methods by which scientists gather this information, one approach compares the intensity of light emitted by a star of unknown distance to the intensity of a "similar" star for which the distance is known. * For instance, suppose that the distance to star X is known and that one wishes to find the distance to a similar star, star Y. Using advanced instrumentation, it is possible to determine the ratio of the intensity of star Y to that of star X. Subsequently, the distance to star Y could be determined if the relationship between light intensity and distance were known.

To some degree, we are able to determine the relative brightness of comparable light sources with our eyes. However, we are unable to quantify measures of intensity. To do so, scientists use phototransistors, devices that generate an electric current that varies according to the intensity of the light source. The SCI Technologies interface uses phototransistors that return a measure of light intensity when directed at a light source.

Fig. 9 depicts an experiment in which students can determine the relationship between distance and light intensity. In this experiment, students construct a model of the initial problem, in which a light-emitting diode (LED) serves as the star and a phototransistor serves as an observer on earth. Students are interested in measuring the intensity of similar "stars" at various distances. To do so, they compare the measured intensity of the LED at different distances from the phototransistor.

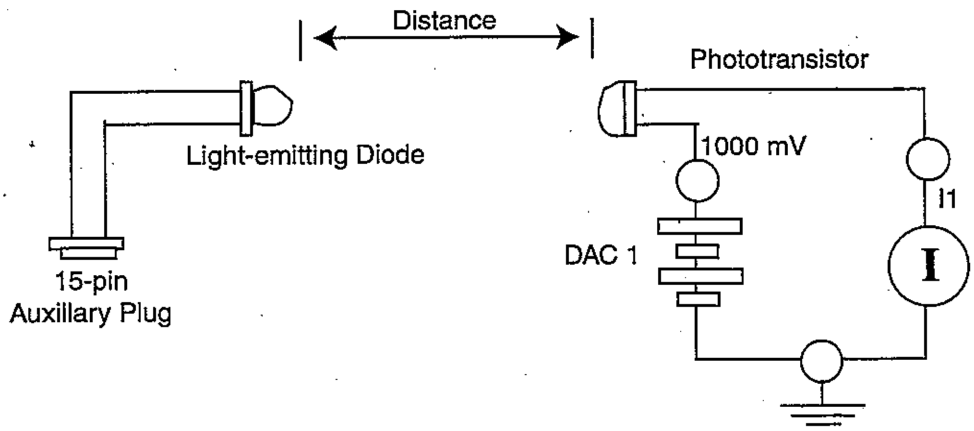

Fig. 9 An Experiment to Investigate the Relationship

* For a more thorough treatment of the techniques by which scientists compute distances in space, the reader is referred to Amend (1969).

Between Distance (to an Observer) and Light Intensity

One problem in this model is background light. Phototransistors do not discriminate between light sources and, thus, readings produced by a phototransistor actually represent the intensity of light emitted by the desired source *and* by all background sources. In order to correct for the presence of background light, students should two readings of light intensity at each distance: one of the background light, and one of the background light and the light emitted by the LED. By subtracting the first measure from the second, a measure of the intensity of the LED for each distance is obtained.

This data can be entered onto a spreadsheet or graphing calculator and, using the analysis features of these devices, students can determine the relationship that best describes the data. Assuming that the data are accurate, students should discover that intensity is inversely proportional to the square of the distance. In terms of the original problem, if star Y is one-fourth as bright as the similar star, star X, then star Y is approximately twice as far from earth as star X.

5. CONCLUSIONS

Although laboratory interfaces are widely used by science educators, the recent emphases on mathematical modelling and integrated mathematics and science have increased their use in the mathematics classroom. Laboratory interfaces represent a low-cost means of collecting research-grade data. They allow students to develop an understanding of mathematics through hands-on experimentation and greatly expand the range of problems that students can investigate. In the process, the use of interfaces enables mathematics students to function as real scientists and modelers.

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Section F

Tertiary Examples

21

The Importance of Student Autonomy in Developing Mathematical Modelling Ability

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SUMMARY

Developing mathematical modelling ability through student-defined problems makes demands of student and tutor which are quite different from those experienced during delivery of traditional mathematics curricula. Defining an area of interest and the problems inherent, along with structuring a written report of their investigation are major difficulties for students faced with this situation for the first time. Through examination of case studies, consideration is given to how these difficulties may be overcome. The preparatory newspaper activity used to prepare students to cope with self-defined and open-ended modelling situations is described in detail and outcomes considered in terms of student interest, motivation and confidence, along with a statistical analysis.

1. INTRODUCTION

The philosophy which underpins all mathematics courses at Edge Hill is that of mathematical modelling and applications, with the overriding aim being that on completing their course, students will have competence and, above all, confidence in these areas. While competence can be planned for and acquired through all aspects of the learning and teaching process, providing appropriate conditions for the development of confidence requires considerable subtlety and sophistication of approach. A student may be adept at concealing insecurity during lectures, seminars, workshops etc. under the guise of shyness, agreeing with the statements of others, or lack of assertiveness. However, it is when given a specific task within a group project or when undertaking a personal piece of coursework which moves into uncharted areas that a student's lack of confidence is most clearly exposed if there has been insufficient or inappropriate preparation to deal with such situations. Clearly, the content of coursework assignments requires extremely careful thought, particularly in respect of progression over the duration of a course.

2. TOWARDS STUDENT AUTONOMY

The first assignment Edge Hill mathematics students can expect to encounter will be highly structured with considerable guidance, not too subtlely implied, as to how to progress towards a solution. The three remaining assignments presented in year one require students to make enquiries, to draw conclusions based on the results of their enquiries and discuss these in the light of the initial situation presented. By the end of their first year students will have encountered four assignments markedly differing in context and the type of mathematics applied. In Year 2, coursework becomes much more open ended with students being required not only to draw conclusions from particular data but to seek out themselves the data required for such deductions. Typical of the assignments are the determination of a mathematical model for the population changes in a town or city of the student's choice, and the identification of a new motorway or village by-pass.

However carefully we feel we have prepared students for being placed in this "semi-open" situation we invariably experience the response of "please tell me what to do!" from a number of students, and not only those we would consider to be weaker mathematically. Success in overcoming these lingering feelings of lack of self confidence is vital if we are to fulfil our overriding aim, for the final year coursework of all students requires them to undertake a "student-defined" modelling project.

The importance of student autonomy in the development of mathematical modelling ability could not be rated more highly, yet the requirement for autonomy generates considerable difficulty for both student and tutor. Although encouraged to consider an area which is meaningful to them, students experience great difficulty generating a willingness to consider something because it is of personal interest. They are aware that it is a mathematics project they are required to undertake and wish to know the mathematics to be used, and the answer, before they start. The difficulties tutors face in encouraging students to consider a problem of interest, and how mathematics can help, tend to be three-fold. Firstly, definition of an area for investigation because it is of interest to the student and not because of the mathematics within that area. If a student is able to define an area of interest by explaining it to the tutor, then the problems contained within that area suddenly come to the fore. Even when a student is aware of the problems contained within an area which is of personal interest, difficulty may arise through an inability to identify them clearly. The final difficulty occurs as a result of students comparing this project with those in other subjects where x thousand words are required. Coming to terms with an indefinite response to their question over length is a major achievement for some students.

An introduction to a student-defined problem is undertaken through an activity which, it is anticipated, will help students feel more confident in relation to the perceived difficulties while ensuring that they are not placed in a straitjacket. From portions of a quality newspaper identified by tutors, students must provide a written explanation of why they found a particular self-chosen article the most interesting. They are also required to suggest how mathematics can help provide a deeper understanding of the situation. Small group discussion of these responses highlights the need for clarity of thinking and faithful transposition of thoughts onto paper. As a consequence, students feel much more secure in the precise definition of an area of interest, the inherent problems, and how mathematics can assist the solution of those problems. Examples from both current and completed projects will serve as illustrations.

Outcomes–Case Studies

One student was immediately attracted to the headline "Steamy novel by GP aged 78 sets literary pulses racing" in The Daily Telegraph of 2nd June 1993 because "it makes a change for national newspapers to carry light stories that are not depressing." The GP had previously had several short stories published but the article highlighted the fact that in this case she was to receive a "£6,000 advance for two novels with an option on a third." This led the student to ponder how much the GP would receive in royalties, and posing the specific question "Do all published books make a profit?" she felt would be of considerable interest. Group discussion of the subsequent development of these areas is based in mathematical economics and initially suggested the study of normal and abnormal profits. While at first interest was focussed on profit and the GP, further discussion led the group to consider the profits made by the GP's publishing company in particular and the whole publishing industry in general.

The next written response which the student shared with the group was that the GP's late brother was Pat Reid, who wrote the best-selling "The Colditz Story". This she had found extremely thought provoking as she wondered whether there might by any similarities in their style of writing. Text analysis, length of sentences and paragraphs were just some of the areas she felt might be worthy of investigation. A lively group discussion followed which included various ideas of sampling, measures of average and dispersion, confidence limits, and a number of other statistical techniques.

During consideration of this article it became clear that the student was actively considering buying the novel referred to in the headline, something that she would not have considered prior to reading the article. An animated discussion followed which analysed a number of situations when the intention of an individual or group suddenly or gradually changed. Factors were considered which brought about these changes and this line of enquiry may subsequently lead to consideration of change of mind involving Catastrophe Theory.

The following day this student, as she did every week, bought the Ormskirk weekly newspaper and was immediately confronted with a two page spread announcing a proposed by-pass to overcome traffic congestion in the town centre. Having already studied cubic splines in the previous year of her course she resolved to find a cubic spline to define the road and then plot this on the map given in the newspaper or one obtained from the local authority. By setting up a coordinate grid and measuring the differences between the proposed route and the cubic spline calculated, she intends to investigate the degree of variance between the two. It is also her intention to measure the curvature at various points along the proposed route in order to identify any curves where speed restrictions need to apply or, alternatively, where roundabouts would be more appropriate.

The newspaper article contains many conflicting comments from local residents, traders, and the local authority on the proposed route and the student saw how this related to the previous day's discussion and the influence the Daily Telegraph article had on her decision to buy the novel. She has, therefore, set out to examine the reactions of the various interest groups to the proposed route in terms of Catastrophe Theory.

Over the four years that we have been using this introduction to student-defined modelling situations this case is probably the one which has demonstrated the most immediate results. The cause and effect happened day-on-day and the newspaper exercise may have subconsciously led the student to think of areas or situations which might form the focus of her project when reading the local weekly newspaper the following day. Ally this to the fact that her project was well advanced in terms of investigative work less than one month after the exercise and one can see that her project was clearly motivated by this activity and arose quite naturally out of personal interest.

Another student who read the same newspaper was attracted to the headline "A false solution to the Doctors' new dilemma" which centred on the fact that some doctors would treat/operate on non-smokers and non-drinkers before those who indulged in these proven health threatening activities. Based on the fact that hospitals do not have a bottomless pit of money to finance treatment this student was moved to ask "what criteria should doctors adopt in establishing a priority list for treatment?" Suggestions from the group included (i) smokers and drinkers; (ii) age, and (iii) cost of the treatment/operation with the discussion progressing to the possibility of devising some form of rating scale. This would be based on how long (a) smokers (b) alcoholics live as within the group there was an awareness that tables exist which predict the life expectancy of people over a certain age.

At this point the discussion suddenly veered off at a tangent as the financing of hospitals became the focal point. The money a hospital has at its disposal was identified as the key issue and suggestions for raising income were considered. In the present economic climate the selling of services i.e. expertise in particular areas of treatment, specific operations, etc. was felt to be paramount. Questions such as "Does the performing of many smaller operations generate more revenue than one large operation?" were discussed at great length. However, the central issue in all the discussion was the need to maximise income and the students' first suggestion for modelling the situation was through a linear programming approach. The next suggestion was based in mathematical economics through study of the sources for delivering a particular treatment and the known demand, in order to identify the optimum asking price for a given service. These could be graphed along with average and marginal revenue as well as average and marginal costs leading to the modelling of the curves by equations.

The discussion now took another interesting turn as the question was raised "How good are these hospitals?" The compiling of League tables as practised in the United States surfaced, quickly leading to various criteria on which these should be based. Recovery rate, death rate, number of beds, and variety of services were just some of the suggestions offered.

The discussion of the articles raised by these two students was both wide ranging, full of mathematical ideas, containing suggestions for enquiry and considerable detail. It is interesting then to note the contrast in the approach to the final year project. This second student returned the following day with a topic totally unrelated to any newspaper article yet she was extremely confident about what she wanted to study – the success rate of top class women tennis players as they progressed through Grand Prix tournaments. In order to do this she intends to give each one a rating for the progress they have made through the tournaments they enter. From this information, time series graphs will be plotted from which phase-space diagrams, which have their base in chaos theory, will be constructed and studied for evidence of a strange attractor. It is then planned to attempt to find a fractal dimension for this system, by the embodied method.

A student who experienced the same introductory activity two years earlier had not studied mathematics beyond the GCSE level prior to entering higher education, yet the day following the introductory activity she announced that having worked the previous summer in an Israeli kibbutz, she would very much like this area to form the focus for her project. It was extremely pleasing to see the confidence with which she embarked on a study of an area of deep personal interest without initially knowing the problems which she might be considering.

Through discussion it became clear that the siting, economy and personnel of a kibbutz were important factors in establishing the success of such a way of life. Consequently she defined the following problems for investigation:

(i) Is there a relationship between the age of the kibbutz and the number of its members?

(ii) Does the size (in terms of population) and distance of the nearest town influence the size of the kibbutz?

(iii) What is the maximum number of kibbutzim that Israel can support?

(iv) Does the interaction between age groups in the kibbutz population mirror that in the state of Israel as a whole?

The first problem was considered statistically and comfirmed the student's own perception that a strong link existed between the age of a kibbutz and the number of its members. Discussion of the kibbutz philosophy leads the student to suggest that there should, perhaps, be a strong positive correlation between the two. However, a number of constraints are identified such as the then current world glut of fruit leading kibbutzim to seek alternative means of income through tourism and light industry.

For the second investigation the country of Israel was divided into equal sized areas and a model developed to predict the number of areas which had a town and kibbutz, a town but no kibbutz, a kibbutz but no town and no town or kibbutz. The observed data provided evidence of validation and Reilly's law provided the basis of a K [=population size/(distance to nearest town)^2] factor for each kibbutz and a KT factor for each town by finding the sum of the K factors for those kibbutzim for which a particular town proved to be the nearest. An examination of the relationship between town population and KT factors suggested a general trend for the KT factor to increase as the size of the town increased. Caution was suggested in drawing conclusions from these figures as the student's personal experience allows greater interpretation and also validation.

In considering the third question, the student made two crucial assumptions: (i) that there will be no change in geographical boundaries (ii) increase/decrease of population is proportional to population size. A population model is developed which gives surprisingly good predictions with an ultimate kibbutzim size of 290. A time-series graph of the growth of kibbutzim shows little uniformity, being related to international pressures on Jews.

The treatment of the final section again demonstrates the student' s considerable perception and ability to apply known mathematics alongside implementation of the modelling process. Her first step is to classify the kibbutzim and Israeli populations into four categories and then establish the flows of people into and out of these categories. Considerable research interviews with representatives of various Israeli information centres and her personal experience produced percentage figures of these flows to one place of decimals. Cascading models were then employed to the current population of each category in order to provide the steady state population predictions. Probably of greatest concern to the Israeli authorities is the downturn in the population of the 0-20 age group, and the student offers perceptive explanations here as well as for the other groups.

3. ANALYSIS OF OUTCOMES FROM THE PREPARATORY ACTIVITY

While these three examples highlight some potential value of the newspaper activity, one must consider a much larger sample before drawing even tenuous conclusions. Consequently, study of all the students who have experienced the activity was undertaken. The first stage was to devise categories for a student's response to the activity and these are shown in Fig.1.

Category	**Response**
1	Demonstrates high confidence in approaching the final year project but the newspaper activity has no direct influence on the subject.
2	The subject of the final year project is directly related to the newspaper activity and is approached confidently.
3	Indecisive and may have several changes of mind.

Category	**Response**
4	Wants to be told what mathematics to use in the project.
5	Dislikes the activity.

Fig. 1 Student Response to the Newspaper Activity

The grades achieved by the students in the various categories were then tabulated (Fig. 2).

(Except for Scores, all figures are percentages)

GRADE (score) / **CATEGORY**	I (5)	II(i) (4)	II(ii) (3)	III (2)	P (1)	F (0)	**Total**	**MEAN SCORE**
1	6	15	6	1			28	3.93
2	1	15	4				20	3.85
3		13	13	1			27	3.44
4		1	12	4	4		21	2.48
5					1	3	4	0.25

Fig. 2 Analysis of Student Grades for the Final Year Projects by Response Category

Finally the numbers in each category over the last three years were tabulated (Fig. 3) in order to investigate recent trends.

Category / **Year**	1	2	3	4	5
1990-91	18		18	64	
1991-92	29	14	38	19	
1992-93	26	23	18	26	7

Fig. 3 Recent Trends

Although this activity was first used with the 1989-90 final year students, by 1990-91 the majority of students still fell into Category 4. However, the next two years see a marked increase in numbers in Categories 1 and 2 suggesting that the newspaper activity might be encouraging students not only to be more confident but also to look for things which are of interest to them. The percentages in Categories 1 and 2 rises from 18, through 43 to 49. Category 3 also rises by more than double, as many Category 4 students, who actively searched for what mathematics they could use, became more open, yet indecisive, in respect of areas of interest to them. Between 1991-92 and 1992-93 the large numbers in this category have moved to either Category 2 or 4, maintaining a general trend for the positive influence of the newspaper activity. While Categories 3 and 4 reduced from 57% to 44% over these two years, a worrying Category 5 group emerged for the first time. However, as there is no indication of students falling into Category 5 this year it is hoped that the ones observed last year was not the start of a trend but rather a chance event.

4. CONCLUSION

The evidence from these statistics and the case studies described, suggests that the newspaper activity has a most positive influence in encouraging students to confidently undertake the study of an area which is of personal interest or concern. While such an approach is still in its infancy, monitoring over the subsequent years will allow refinement of the present approach and the introduction of more appropriate initial activities will, it is hoped, develop the twin areas of student confidence and autonomy when undertaking modelling activities.

22

Teaching Mathematics to Biologists –Some General Aspects and Modelling Examples

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SUMMARY

This is a report on some ideas and experiences of my teaching mathematics to diploma students of biology at the University of Heidelberg. The examples deal with a basic model of enzyme kinetics, the Michaelis-Menten model for the food intake, and an application of the Michaelis-Menten Law to find a predator-prey system modelling the saturation of the predator. The emphasis is on how these models can be handled by elementary concepts of calculus and explained to students in the life sciences.

1. THE STARTING SITUATION

Actually the application of mathematical methods in biology is developing very quickly. Therefore, since the winter semester 1985/86 the diploma students of biology at the University of Heidelberg have to take either mathematics or alternatively physics as a subsidiary subject. The mathematics course takes place during their first three semesters, four hours a week lectures and two hours exercises in small groups. The students of biology also have to take a beginner's course on a computer language (Pascal, Fortran or C). Each year there are about 120 beginners in the diploma study of biology and 60 to 90 decide to take mathematics.

Because of the complexity of biological processes, complex mathematical theories are often necessary in mathematical modelling. On the other hand, only a few students of biology have a good understanding of mathematics from high school. From school time on they usually have to first surpass a threshold before doing mathematics. So the question is: Which way should a teacher take?

- Should one take the descriptive view without explaining mathematical details?
- Should an understanding of mathematics also be taught?
- Is it wise to insist first on mathematical technics and main theorems that seem to be indispensable for a basic mathematical education?

My opinion from the very beginning was that one should teach an understanding of what is mathematically going on, because only then can a biologist apply the mathematics correctly, for example decide correctly which statistical test has to be used. My purpose was not only to describe some mathematical examples from biology but to teach the students to work on their own with the methods and technics from calculus, linear algebra, probability and statistics. But I had the idea that what I took for the indispensable part of mathematical basic concepts should not be taught by insisting on them, but by showing how useful they are and which interesting results one can get, if one has an understanding of mathematics and its applications.

2. TEACHING AND MODELLING

I soon learned that it is very important for successful teaching to explain the mathematical notions as examples of biology. Very often it is even possible to describe first a biological model and then abstract the mathematical concepts. Here I learned that the level of abstraction should be taken as low as possible. Then the biology students were very well motivated and all ears during the lectures, had good learning success and were encouraged to surpass the mentioned threshold. In this sense modelling has turned out to be a very good means in the mathematical education.

The next important point I learned was to choose the right models to explain mathematical notions. There are examples which explain in a very clear way a mathematical notion from the mathematicians' point of view. Such examples did only prove suitable if they did fit into the biological reality. A biologist automatically thinks also about the biological meaning and if he finds a biologically irrelevant assumption, his interest in the mathematical notion decreases. In this way we can not reach our goal.

In a large part my teaching was directed towards learning mathematical methods to see general aspects, ideas and concepts. With the learned mathematical concepts one could then go and try to explain other even more interesting models or find new ones. I myself was wondering which interesting models and applications could be handled by elementary concepts, for example elementary curves like parabolas or hyperbolas.

The mathematical terminology must be translated into the terms that are used in every day life or in the biological world. For example the geometric illustration of the differential quotient by the increment of the tangent is not enough. The biologist who sits in front of his culture of bacteria sees at most a tangent to the bowl in which the bacteria are cultivated. Nevertheless we may not forget the tangent but we must take more time to explain the connection between differential quotient, tangent and growth-rate of a population. This passage from a mathematical notion to its geometrical illustration and to its meaning in the biological world can easily be found in the diagrams that are drawn of the mathematical modelling process. A consequent use of the model conception of the relationship between mathematics and other subjects, as it was presented for example at ICTMA-6, will certainly help to improve teaching mathematics to nonmathematicians.

There is a great number of publications on the topic "Mathematics as a Service Subject" (see Blum (1988), Howson e.a. (1988)). Unfortunately they are unknown to many teachers in Germany. My approach corresponds, in many ways, to the educational discussion in the literature and has put some of the statements into practice.

The following two models that have been analysed by complex mathematical theories can be handled by elementary concepts from calculus and linear algebra.

3. THE MICHAELIS-MENTEN MODEL

The Michaelis-Menten model handles the intake of food by the digestive organs. The nutrient substratum S gets transformed by an enzyme reaction into a product P that can be taken in by the body. The investigation leads to the following initial value problem, where b_0 and b_1 are positive constants:

$$(1)\quad \dot{u} = -u+(u+b_0)v, \quad \dot{v} = \frac{1}{\epsilon}(u-(u+b_1)v), \quad u(0) = 1 \text{ and } v(0) = 0$$

$\varepsilon = e_0/s_0$ is the quotient of the enzyme concentration by the substrate concentration at the beginning of the process, a small quantity in practice.

A result of singular perturbation theory (Murray (1989), p.110 ff) says: If ε is small enough then each point $(u(t), v(t))$ of the orbit is arbitrarily close to a hyperbola H_{b_1} from a certain time t_0 on. H_{b_1} is given by $v = \frac{u}{u+b_1}$.

A computer simulation shows this fact for some solutions with different initial conditions. The following computer drawings are done for $\varepsilon = 1$ and $\varepsilon = 1/3$, $b_0 = 1$ and $b_1 = 3$. The hyperbola H_{b_1} is the curve marked with little lines and dots.

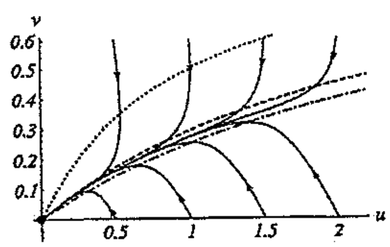

Fig. 1: Phase portrait for $\varepsilon = 1$

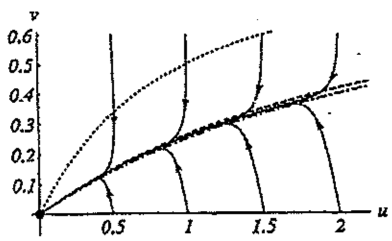

Fig. 2: Phase portrait for $\varepsilon = 1/3$

H_{b_1} is the isocline of horizontal orbit directions. It can be calculated that there is another isocline-again a hyperbola- H_b with the following properties:

- H_b is arbitrarily close to H_{b_1}, if ε is small enough. (See Fig. 2, where $\varepsilon = 1/3$)
- From some time t_0 on the orbit of our initial value problem remains between H_{b_1} and H_b and thus remains arbitrarily close to H_{b_1}, if ε is small enough.

One might suppose that H_b is the vertical isocline. This is not the case. In Figs. 1 and 2 the vertical isocline is the uppermost dotted hyperbola and H_b is the dashed hyperbola between the vertical and the horizontal isocline.

This result is very well applicable in practice because the enzyme concentration is small with respect to the substrate concentration. ε lies between 10^{-7} and 10^{-2}. This means practically that after the process has got going (from the mentioned time t_0 on) we have $v = \frac{u}{u+b_1}$. Lets denote the concentrations of the product P and the substate S by lower case letters p and s. The result expressed in terms of p and s is the

Michaelis-Menten Law:

The intake rate $\dot{p}$ of the product and the consumption rate $-\dot{s}$ of the substrate are equal and depend on the concentration s of food according to the following formula:

$$\dot{p} = -\dot{s} = \frac{as}{s+B}, \quad a \text{ and } B \text{ positive constants}$$

a is the supremum of the rate of intake and B is called the *Michaelis-Menten Constant.* An interpretation of B is the following: If the concentration of the substrate has the value $s = B$ then the rate of intake is just half the supremum of the rate of intake.

4. A PREDATOR-PREY MODEL

A predator-prey system can be modelled by a system of differential equations for the sizes x of the prey population und y of the predator population in the following way:

$$\dot{x} = xG(x) - J(x)y, \quad \dot{y} = y(-D(y) + P(x)) \tag{3}$$

For the specific growth rate $G(x)$ of the prey in absence of predators we use the well known logistic model $G(x) = p(1 - \frac{x}{K})$, $p > 0$ and $K > 0$.

The specific death rate $D(y)$ of the predators in absence of prey is modelled by a constant $D(y) = d$. We regard the prey population as the substrate in the sense of the Michaelis-Menten model. For the intake rate of one predator we use the Michaelis-Menten Law $\dot{p} = \frac{ax}{x+B}$ and put the specific growth rate $P(x)$ of the predator population with respect to the intake of food equal to the intake rate, $P(x) = \frac{ax}{x+B}$. Observations in nature suggest that predators chase only as long as they are hungry. Their saturation depends on the other hand on how they can intake their food. To model saturation we put the chase rate $J(x)$ of one predator equal to the intake rate, $J(x) = \frac{ax}{x+B}$. Thus we end up with the following predator-prey model:

$$(4) \qquad \dot{x} = px\left(1 - \frac{x}{K}\right) - \frac{axy}{x+B} \quad , \quad \dot{y} = -dy + \frac{axy}{x+B}$$

The model exhibits mainly three possible behaviours:

- *No coexistence*
- *Coexistence in a stable steady state*
- *Coexistence in a stable oscillation.*

The case of no coexistence is shown in Fig. 3. If the capacity K of the prey is too little then the predators will die out while the prey population approaches its capacity. This can be interpreted in such a way that there is not food enough to guaranteed the predators' survival. "Too little" means in the mathematical model $K \leq \lambda := \frac{Bd}{a-d}$. See Fig. 3, where $K = 3$ and $\lambda = 3.9$. The other parameters in Fig. 3 to 5 are $d = 3, a = 4, B = 1.3$. Fig. 4 demonstrates the possibility of coexistence in a stable steady state for $K = 8$. Coexistence in a stable oscillation is observed for $K = 10$ in Fig. 5. The closed stable orbit corresponding to the stable periodic solution is printed fat.

Many of these facts can be explained using only elementary facts from linear algebra and calculus (see Riede (1993) and (1994)). The most efficient tool is the concept of the direction field. One aspect of the direction field is indicated in the figures, namely the horizontal isocline, the straight line, $x = \lambda$, marked with little lines and dots. The vertical isocline, a parabola marked with dots, is also plotted.

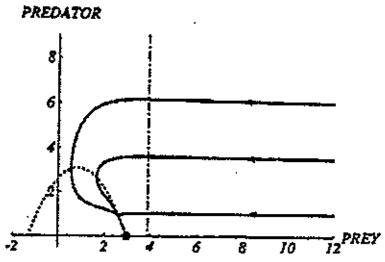

Figure 3: No coexistence

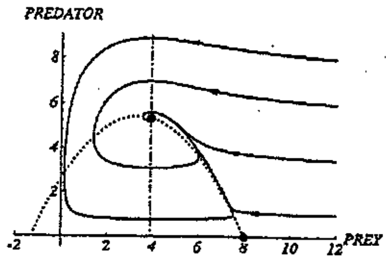

Figure 4: Stable steady state

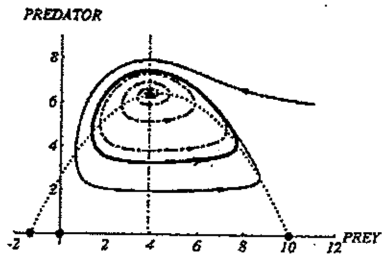

Figure 5: Stable oscillation

After these wonderful mathematical results have been explained biologists ask if an observed oscillation in nature is governed by such a model. But this is not easy to decide. However experiments in the laboratory affirm its relevance (see Waltman (1983)). One could conclude the lectures on these models by showing Waltman's diagrams of experimental data.

5. REFELECTIONS ON TEACHING THE MODELS

The point of this explanation of the Michaelis-Menten Law is the calculation that shows that nearby the hyperbola H_{b_1} there are other hyperbolas which are crossed by the orbits from above (see Riede 1994). In my lectures I often carried out such technical details, if I found it useful for the applications and if the technical apparatus did not surpass a certain amount. My experience was that if the calculations are done slowly step by step and clearly arranged, then a good learning success is possible such that students learn to do such calculations on their own. They learn which possibilities are given for manipulating and transforming a formula and that one has to do this carefully to obtain correct results. This is also very important if they are going to tell a computer to do the calculations in a more complex situation.

Computer plots, like those above, proved a very good way to illustrate mathematical notions and results. Similar to the differential quotient we must take some time to explain the speed vector of an orbit, its geometric interpretation as an arrow tangent to the phase curve and its connection to the growth or decrease rates of both species. The students are also better motivated if a formula is explained by a nice picture. But there is also a danger: It can occur that to the teacher the figure shows a certain feature but this feature is not as obvious to the students, especially if the biological meaning is not obvious. The attention that a nice picture draws can be used as an excellent opportunity to explain again the mathematics and the biology behind the figure.

Concluding I like to say that I hope that these investigations on relatively realistic examples will augment the teachable and learnable possible contents of a mathematics course for biology students.

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Mathematical Modeling In Higher Distance Education

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SUMMARY

The Open university of the Netherlands (OuNL) caters for the educational needs of what can be most succinctly described as non-traditional students. This paper's first section sketches a general picture of the OuNL. Section 2 then fills in some details on its Natural Sciences programme. This programme features a modelling course called *Mathematical modelling for life scientists*, which is briefly introduced in section 3. Some typical aspects of teaching mathematical modelling in a OuNL setting are highlighted in the last two sections. OuNL education, being distance education, calls for alertness to students' difficulties in formulating differential equations (section 4). And its being higher education warrants a rather in-depth treatment of modelling (section 5).

1. THE OPEN UNIVERSITY OF THE NETHERLANDS

With the formation of the Open university of the Netherlands (OuNL) in 1984, the country saw the introduction of a new form of higher education, at both university and higher vocational levels. OuNL education is open to anyone aged 18 or over; no other formal qualifications are required. Instruction is mainly through printed materials, allowing study to take place at home, in students' own time, at their own pace. Tutoring facilities, from individual consultation to group sessions in one of 18 study centers, are available optionally. OuNL education is open *higher distance* education.

OuNL courses are self-contained units, generally requiring 100 hours of study each. They can be combined in various ways, basically allowing students to choose between four possibilities: a single course, a free combination of courses, a short academic degree programme and a full academic degree programme leading to the degree of 'doctorandus', roughly the equivalent of a masters degree, which at traditional Dutch universities is obtained upon completion of a 4-year full-time programme.

Enrollment is currently over 50,000. The student population is an extremely heterogeneous one in terms of age, previous education and professional experience, to mention but a few categories. However, a global distinction can be made between two subpopulations. About 40% of OuNL students have no formal qualifications for enrollment at traditional institutions for higher education. The OuNL offers them a *second chance.* In fact, doing just this was one of the major goals set for the OuNL upon its foundation, fitting in with a more general equal opportunities policy.

The remaining 60% of OuNL students appear to be well-educated people who wish to broaden their existing knowledge and skills and prefer the conditions of studying at the OuNL to those of traditional higher education. The OuNL is said to offer them a *second way.*

2. STUDYING NATURAL SCIENCES AT THE OPEN UNIVERSITY

The OuNL consists of seven faculties, representing the following fields of study: law, economics, management and administration, technology, social sciences, cultural sciences, and natural sciences.

The study of natural sciences at the OuNL attempts to integrate four basic disciplines: biology, chemistry, geology and physics. Some individual courses are discipline-oriented while others are oriented towards integration. All degree programmes are multidisciplinary in approach.

A second characteristic of the programme offered by the OuNL's

Department of Natural Sciences is its being geared towards linking the understanding of natural phenomena to the analysis of problems of policy and management. Some courses specifically aim at achieving this kind of integration. Obviously, developing courses like these is not feasible without contributions from social scientists. For that reason, social scientists are on the Natural Sciences staff and among the external specialists to whom the authoring of (parts of) courses is commissioned.

The dually integrated framework outlined above is applied to two fields of study. One is the environment. The second, distinct from but not unrelated to the first, is the field of nutrition and toxicology.

3. MATHEMATICAL MODELING IN HIGHER DISTANCE EDUCATION

The OuNL's Natural Sciences programme features a modelling course called *Mathematical modelling for life scientists.* Its first part is devoted to learning to speak the language of mathematics, rather than rely on the mathematical bag of tricks. One is taught how to formulate mathematical models of dynamic processes, and how to study their behavior both analytically and numerically (through the use of computers). The second part is devoted to a variety of applications to the life sciences: chemical kinetics, physiological control, growth, population dynamics and fishery. A textbook (Doucet and Sloep, 1992), an accompanying workbook (Sloep *et al.*, 1992) plus software (*Stella* II from High Performance Systems, 1992) comprise the course materials.

Some typical aspects of teaching mathematical modelling in a higher distance education setting are highlighted in this paper. OuNL education being distance education calls for special didactic techniques (section 4). And its being higher education warrants a rather in-depth treatment of modelling (section 5).

4. DISTANCE EDUCATION: THE PROBLEM OF FORMULATING DIFFERENTIAL EQUATIONS

It is natural for printed material, designed for use in distance education, to contain didactic features usually not to be found in traditional textbooks. Some of the more prominent ones are listed in Box 1. They feature in most OuNL courses, including *Mathematical modelling for life scientists.*

Box 1. Didactic features of printed material in distance education

- A typical study unit opens with study instructions (or, less severely, suggestions). These include a list of learning objectives and an estimate of the number of study hours required for the unit.
- Key terms and concepts are indicated in both the running text and the left hand margin.
- The left-hand margin also features so-called 'margin texts', serving as informal comments and hints. They can be thought of as the sort of things a live teacher could have said at that point in his or her lecture.
- In-text-questions are meant to make a student pause and in some way or another process the information contained in the preceding text. In-text-questions are immediately followed by an answer.
- Self-assessment questions appear at the end of a study unit. They enable students to check whether key concepts have been mastered.

Besides these general ones, subject-specific didactic features may be called for. In *Mathematical modelling for life scientists* the skill of formulating differential equations is a case in point.

Even in non-distance education many students find it difficult to formulate a model's behavioral equations on the basis of information presented mainly as text. Very often, they have to be carefully guided by their teacher. It follows that in a distance education modelling course, with students having no teacher around when they get stuck, this step is a matter of serious concern. It was found that one way to at least partly overcome the problem is to have students visualize the information they have to work from. This visualization is standard procedure in the simulation software that comes with the course. It will be illustrated by an example, which concerns the physiological regulation of glucose in the blood by the hormone insulin. As an introduction, Box 2 contains a slightly reworded quotation from Doucet and Sloep (1992).

Box 2. A model of blood glucose regulation by insulin
We shall inspect the regulation mechanism by means of a simple model taken from McClamroch (1980). It contains two state variables: the insulin level in the blood and the blood glucose level; and one or two input variables: the feeding regime, and possibly (for diabetes patients) an injection regime for intravenously administered insulin.

- Glucose enters the blood from the digestive tract by hydrolysis of carbohydrates in a meal (an input variable).
- Glucose also enters the blood by the breakdown of liver glycogen in response to low blood glucose levels. Breakdown rate depends on the blood glucose level and is assumed not to depend on the amount of glycogen present in the liver.
- Glucose leaves the blood by conversion to glycogen (actually, passage into tissue cells followed by conversion), at a rate depending on both insulin level and glucose level.
- Insulin is synthesized in certain areas of the pancreas, and is released into the bloodstream at a rate which depends on the blood glucose level. A second, optional, source of insulin is injections (an input variable).
- Insulin does not remain in the bloodstream, being degraded by an enzyme (insulinase), at a rate depending on blood insulin level.

Here, without entering into too much detail, the role of Stella modelling and simulation software in the initial stage of the modelling process will be discussed. Fig. 1 shows the so called Stella diagram built on the basis of the five model assumptions formulated above. It is of interest to note that building the diagram is a stepwise procedure, with each one of the model assumptions corresponding to one of the flows in the diagram.

Now, starting from the diagram, formulating the model equations is relatively straightforward:

$$x' = F_1(x) - F_2(x, y) + u_1/V \qquad (1)$$
$$y' = G_1(x) - G_2(y) \qquad (2)$$

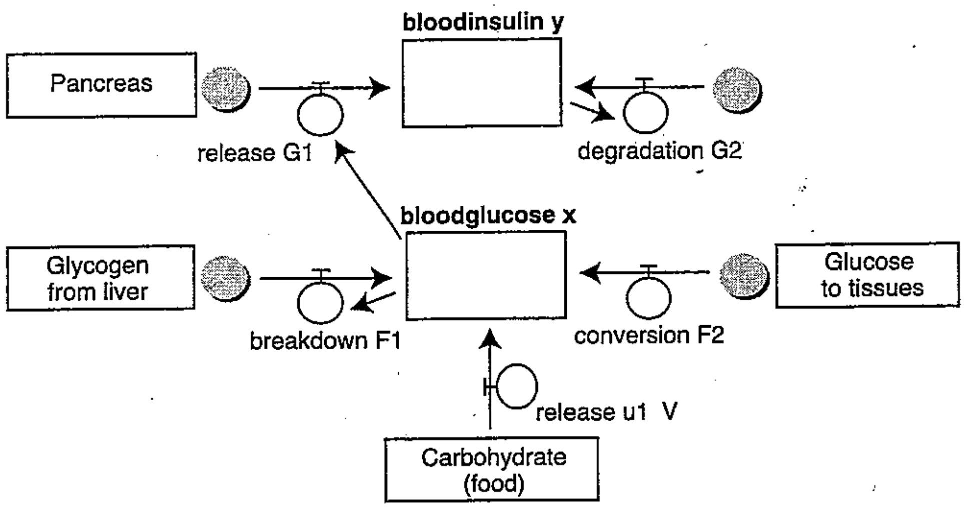

Fig. 1. Stella diagram for blood glucose regulation by insulin.

According to students, the gap they feel at first between text and equations is effectively bridged via the visual intermediary foothold provided by the diagram. The notoriously difficult step of formulating a model's behavioral equations turns out to be enormously facilitated by first constructing the diagram. In fact, this is one of the reasons why this particular species of software has turned out to be so very well-suited for mastering the art of modelling through distance education.

Of course equations (1) and (2) are as yet unsufficiently specified. But, from a distance-educational point of view, the worst is over. Having got this far, students confidently proceed by feeding additional information into the equations (specific functional relationships, parameter values). The software can then again be relied upon to guide them through the process of actually running simulations and producing graphs like Fig. 2, showing the bloodglucose peak after a meal in a diabetes patient who receives no extra insulin.

5. HIGHER EDUCATION: MATHEMATICAL MODELS AS CARRIERS OF SCIENTIFIC KNOWLEDGE

In higher education, whether or not it is of a distance nature, a natural sciences course on modelling should not restrict itself to the fairly

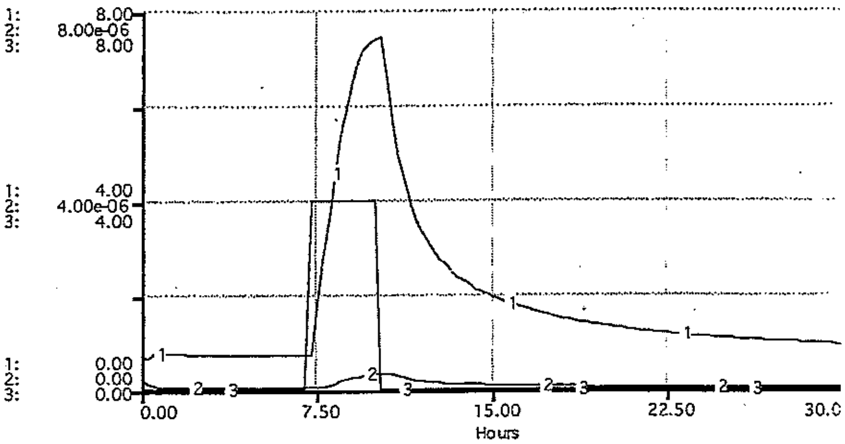

Fig. 2. Simulation output.
Diabetes, no extra insulin, one meal.

1: bloodglucose x. 2: bloodinsulin y. 3: glucose release u_1/V from food carbohydrate.

practical level of building models and numerically implementing them through the use of a simulation *package*.

Students of *Mathematical modelling for life scientists* are treated to a rather in-depth exposition on the nature of models, their role in scientific research, and the justification for their use. A central part of this exposition concerns the so-called semantic view of theories. This paper's final section is devoted to this issue.

The Semantic View of Theories

The semantic view of theories revolves around three key notions: natural system, theoretical model and theoretical hypothesis. A *natural system* is a part of the empirical world. It has spatial and temporal dimensions. In a particular case, what makes up a natural system is determined by what one is interested in . Studying the natural system gives rise to ideas of the way it might operate. A *theoretical model* is the formalization of these ideas in mathematical language.

The relationship between natural system and theoretical model is a subtle one. It is obvious that the theoretical model is inspired by the natural system. But logically speaking there is no connection whatsoever between the two. The theoretical model makes no claims about the behavior of the natural system. It does not make claims about anything in the empirical world, for that matter. Still, making claims about the natural system is what the modeler ultimately aspires to. The point to be made here is that the theoretical model is not the proper vehicle for such claims. Allowing such claims to be made is the task of the third ingredient of the semantic view of theories, the *theoretical hypothesis.* It simply says: 'the theoretical model is a model of the natural system'. The theoretical hypothesis is a contingent statement, its truth being contingent upon the empirical world of which the natural system forms part. The theoretical hypothesis could turn out to be false. Without it, a theoretical model cannot inform about the empirical world. Put differently, without a theoretical hypothesis the theoretical model cannot perform the task of being a carrier of scientific knowledge.

Twin Lakes

The three notions introduced above will now be applied to an example from Doucet and Sloep (1992). The example, called *Twin Lakes,* is introduced in Box 3.

Box 3. Twin Lakes
Twin Lakes is a system consisting of a river passing through two lakes. Let us assume that, by accident or negligence, a load of noxious chemical enters the first of the two lakes. The accident calls for practical measures, one of which is that no swimming is allowed in the lakes as long as the concentration exceeds a certain value. The holiday resorts along *Twin Lakes* will have to be notified that swimming will be forbidden during some period of time. The people concerned–hotel owners, other business people, and their customers –would like to know how long this period will last.

Natural System

Obviously, the natural system is the pair of lakes called Twin Lakes, shown in schematic top view in Fig. 3.

Theoretical Model

The problem can be solved by using a two-compartment model with two coupled differential equations. This pair of equations is what the theoretical model consists of. The model will be referred to as: the

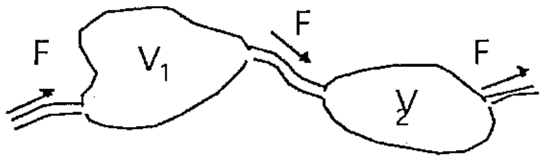

Fig. 3. Washout of a chemical from *Twin Lakes*.
F: flow. V_1: volume of lake 1. V_2: volume of lake 2.

two– lake cascade system. In the language of the semantic view of theories, this theoretical model can be defined as follows.

A theoretical model is a two-lake cascade system if and only if:

$$C_1'(t) = -\frac{F}{V_1}C_1(t),\ C_1(0) = \frac{Q_0}{V_1}$$

$$C_2'(t) = -\frac{F}{V_1}C_1(t) - \frac{F}{V_2}C_2(t), C_2(0) = 0$$

Theoretical Hypothesis

The theoretical hypothesis says: the two-lake cascade system *is* a model of *Twin Lakes*. Or, equivalently: *Twin Lakes* is a two-lake cascade system. The theoretical hypothesis provides a link between theoretical model and natural system. This link has the status of a claim. The theoretical hypothesis claims that the natural system called *Twin Lakes* has the property of being a two-lake cascade system. What this property amounts to is stipulated by the definition of the theoretical model.

The theoretical hypothesis can also be phrased in a more elaborate form. It then consists of a conjunction of contingent statements as listed in Box 4.

Box 4. ***Twin Lakes*** **theoretical hypothesis**

1 C_1 represents the concentration of the chemical in lake 1.

2 C_2 represents the concentration of the chemical in lake 2.

3 F represents the flow rate of the water.

4 V_1 represents the volume of the lake 1.

5 V_2 represents the volume of the lake 2.

6 The values of V_1 and V_2 are constant.

7 The value of F is constant.

8 The amount of chemical dumped is negligible with respect to the amount of water; strictly speaking, it should be 0 (otherwise the volume of lake 1 changes).

9 Mixing between chemical and water is immediate and complete (otherwise we need a more than two compartment model).

10 All chemical in the lakes is dissolved in the water (no binding to the sediment).

An important distinction must be made between statements 1-5 and statements 6-10. Statements 1-5 are identifications. The use of the verb 'represent' is typical of identifications. Statements 6-10 are assumptions. This version of the theoretical hypothesis is equivalent to the more succinct version: the various items in the list jointly make up the theoretical hypothesis. They are all, both identifications and assumptions, contingent statements, and may be false.

It is not common among modelers to speak of theoretical model *and* theoretical hypothesis. However, explicitly adhering to the distinction between the two has the advantage of enhanced conceptual clarity. This applies in the stage of formulating identifications and assumptions, when the distinction makes one aware that these form part of the theoretical hypothesis, not of the theoretical model. In the sequel, it will be shown to apply to testing. Testing a theoretical hypothesis, that is.

Testing the Twin Lakes Theoretical Hypothesis

In order to be able to carry out a test of the theoretical hypothesis some preliminary work has to be done. First, the system of coupled differential equations that, together with a specification of the state variables, comprises the theoretical model, has to be solved. The solutions are:

$$C_1(t) = \frac{Q_0}{V_1}e^{-\frac{F}{V_1}}$$
$$C_2(t) = \frac{Q_0}{V_1 - V_2}\left(e^{-\frac{F}{V_1}t} - e^{-\frac{F}{V_2}t}\right)$$

Second, some numerical values have to be specified. Since **Twin Lakes** is not an authentic example, these are of an arbitrary nature and are given without discussion: $V_1 = 1,000,000m^3, V_2 = 600,000m^3, F = 50,000m^3$ per day, and the amount of chemical dumped into lake 1 is 500 kg. Inserting these values into the solutions yields

$$C_1(t) = 0.0005^{-0.05t}$$
$$C_2(t) = 0.00125(e^{0.05t}i - e^{-0.083t})$$

Now, on the basis of the theoretical model a prediction can be made. Since a prediction is about some point in time, as a final specification a value of t must be chosen. The prediction will be about $t = 30$ days, and will focus on the second state variable, $C_2(t)$. From the equation it is found that $C_2(30) = 1.75$. This implies that at $t = 30$, the chemical in lake 2 will have a concentration of 1.75 $\mu g/L$. Two things may occur. Either the concentration at day 30 is not equal to 1.75 $\mu g/L$, or it is. In other words, either the prediction does not bear out, or it does. (Of course, in reality things will not be that clearcut. For instance, at $t = 30$, the concentration could turn out to be 1.74 $\mu g/L$. However, the subject of statistical variation will not be dealt with here.)

The logical argument involved in testing a theoretical hypothesis takes on a different form depending on whether or not the prediction bears out. In case it does not, the argument looks like this, with the theoretical hypothesis denoted as Th, the prediction as P:

- premiss 1 — if Th then P

- premiss 2 not-P
- conclusion not-Th

In the language of logic, the argument is a *modus tollens*. A modus tollens is a valid deductive argument. Here, it results in *falsification* of the theoretical hypothesis.

In fact, a next step ought to be added to the argument. It was stressed that the theoretical hypothesis is a conjunction of contingent statements. In the case of *Twin Lakes*, the number of contingent statements is ten. Taking this into account, the theoretical hypothesis can be written as (with '1' for 'statement 1', etc.):
$Th =$ (1 and 2 and 3 and 4 and 5 and 6 and 7 and 8 and 9 and 10)

And its negation:

not-$Th =$ not-(1 and 2 and 3 and 4 and 5 and 6 and 7 and 8 and 9 and 10)

Through an argument called *negation of a conjunction* it can now be concluded that

(not-1 or not-2 or not-3 or not-4 or not-5 or not-6 or not-7 or not-8 or not-9 or not-10)

To sum up, here is the complete argument:

- premiss 1 if (1 and 2 and...and 9 and 10) then P
- premiss 2 not-P
- conclusion 1 not-(1 and 2 and ...and 9 and 10)
- conclusion 2 (not-1 or not-2 or 4 ... or not-9 or not-10)

This conclusion is to be interpreted as an invitation to go and look for which statement is false. Or, better: which statements are false ('or', as it is used in conclusion 2, is of the inclusive type).

Although in principle one or more of the identifications could be false, often the assumptions are to blame. The different ways in which assumptions may be mistaken will not be dealt with here. However, a related issue should be stressed: the list (statements 6-10) is by no means complete. To make matters worse, there is no truly satisfactory way of making it complete. And yet one needs the list to be complete for only then the theoretical hypothesis guarantees that the theoretical model is a model of the natural system under study.

One could expand the list to include such assumptions as:

11 The chemical does not break down.
12 Any chemical leaving lake 1 immediately enters lake 2 (implying that the stretch of river between the lakes has zero length).

But adding these does not result in a list that can be considered complete. Factors not included simply because they are unknown to the modeler may affect the theoretical hypothesis truth. The only way out is adding the assumption: no other disturbing factors are operative. The list is complete. But a new problem arises: there is no way of guaranteeing the truth of this blanket assumption.

In case the prediction does bear out, the logical argument can no longer be deductive but has to be inductive, and is slightly more involved as well. It will not be presented in this paper; the reader is referred to Doucet and Sloep (1992), where it is discussed in some detail. Here, instead of the proper argument, a potential trap will be dealt with, a trap frequently fallen in to by modelers (see Oreskes *et al.*, 1994). The argument then runs like this:

- premiss 1 if Th then P
- premiss 2 P
- conclusion Th

In the language of logic, this argument is called *affirming the consequent.* Affirming the consequent is an invalid argument, a logical fallacy. Its use results in a 'verification' which is totally unfounded. The logical crux of the matter is that, given premiss 1, premiss 2 being true is very well possible without Th being true. Or, in modeling terms: if the prediction bears out, some other mechanism than the one incorporated into the theoretical hypothesis may be behind it. And, finally, in terms of *Twin Lakes*: if, after 30 days, the concentration in the second lake is equal to 1.75 $\mu g/L$, this does not prove that *Twin Lakes* is a two-lake cascade system. What a prediction bearing out does provide is a more modest confirmation of the theoretical hypothesis.

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An Attempt to Integrate Traditional Applied Mathematics and Modern Mathematical Modelling Activities

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" Men must be taught as if you taught them not,
And things unknown proposed as things forgot."

Alexander Pope
Essay on Criticism

SUMMARY

Since the pedantry of traditional applied mathematics prevalent this century, with its insistence on involved algebraic manipulations and sophisticated calculus, has done little to enhance the ordinary student's understanding of real world problems, the possibility of the construction of some form of integrated course was investigated.

This article is a report of an attempt to develop such a non-specialist modelling course based firmly on traditional applied mathematics to which has been added the essential ingredient of experimentation.

The content and organization of the course is described, with an emphasis placed on the novel methodology developed and some of the specific activities undertaken by the students, and the issues relating to the assessment of the students' work.

1. INTRODUCTION

The present position of the teaching of traditional applied mathematics in the English educational system can be said to have its origins in the early years of the nineteenth century. The establishment of the Analytical Society in Cambridge, which was followed closely in 1819 by the permanent association to be known as the Cambridge Philosophical Society, resulted in widespread changes in the examination system at Cambridge (Ball, 1889).

The character of the instruction in mathematics at the university was largely dependent on the textbooks then in use and the rule that an examination question should not be set on a new subject in the tripos examination unless it had been discussed in some treatise suitable and available for Cambridge students was invariably accepted. The textbook writers would include the more recent examination questions suitably modified in their textbooks and thus the demand for new questions led to more and more difficult questions. To promote their cause, the Analytical Society issued in 1820 two volumes of examples illustrative of the new method. Furthermore Whewell's Mechanics (1819) and Dynamics (1823) appeared at that time (Earnshaw, 1845). By 1830 the analytical methods had almost completely superseded the fluxional and geometrical methods in the rest of the country.

By the beginning of the twentieth century we find widespread dissatisfaction with the whole position of the teaching and assessment of applied mathematics. At the International Congress of Mathematicians meeting in Rome in 1908 a proposal of Dr David Eugene Smith, Professor of Mathematics at Columbia University, New York resulted in the appointment of an International Commission on the Teaching of Mathematics. The immediate cause which led to the formation of the International Commission was the divergence between mathematical and pedagogic requirements in the schools and universities. Its brief was to secure a series of reports on the state and progress of mathematical instruction in various countries of the world.

In these reports (Board of Education, 1912) we find such typical sentiments as

"They are written solely with a view to examinations. The authors aim at brevity in book work and the omission of everything that is not likely to be asked for in an examination. They deliberately avoid any suggestion of anything beyond the syllabus of certain examinations. The examples they work out in the text are selected for their difficulty, not to illustrate any principle."

and

Another serious defect, which has been generally attributed to the system of the Mathematical Tripos, is the almost complete divorce between mathematics and experimental physics.

Although the mathematician has given about half his time to "Applied Mathematics" he need have, and frequently has had, no knowledge of experimental physics. Normally, he goes to no experimental lectures, he does no work in a laboratory, and the experimental facts which he learns in his mathematical textbooks are usually of the simplest character, reduced to an abstract and almost conventional form, suitable for the direct application of mathematical analysis."

2. THE MODELLING DIAGRAM

Every serious teacher engaged in mathematical modelling has, from time to time, either used the now classic seven box diagram (Penrose, 1978) or constructed a diagram for the methodology appropriate to the level of instruction and the class of problems under consideration. The approach adopted in the course described here demanded its own diagram to illustrate the loose connection between traditional applied mathematics and the essential experimentation. (Fig. 1). Several interdependent loops can be identified:

(A) the *real world-problem-solution* loop
 This is the basic loop and it is this loop, in which a real world problem is identified and a solution requested, that determines the complete modelling activity.

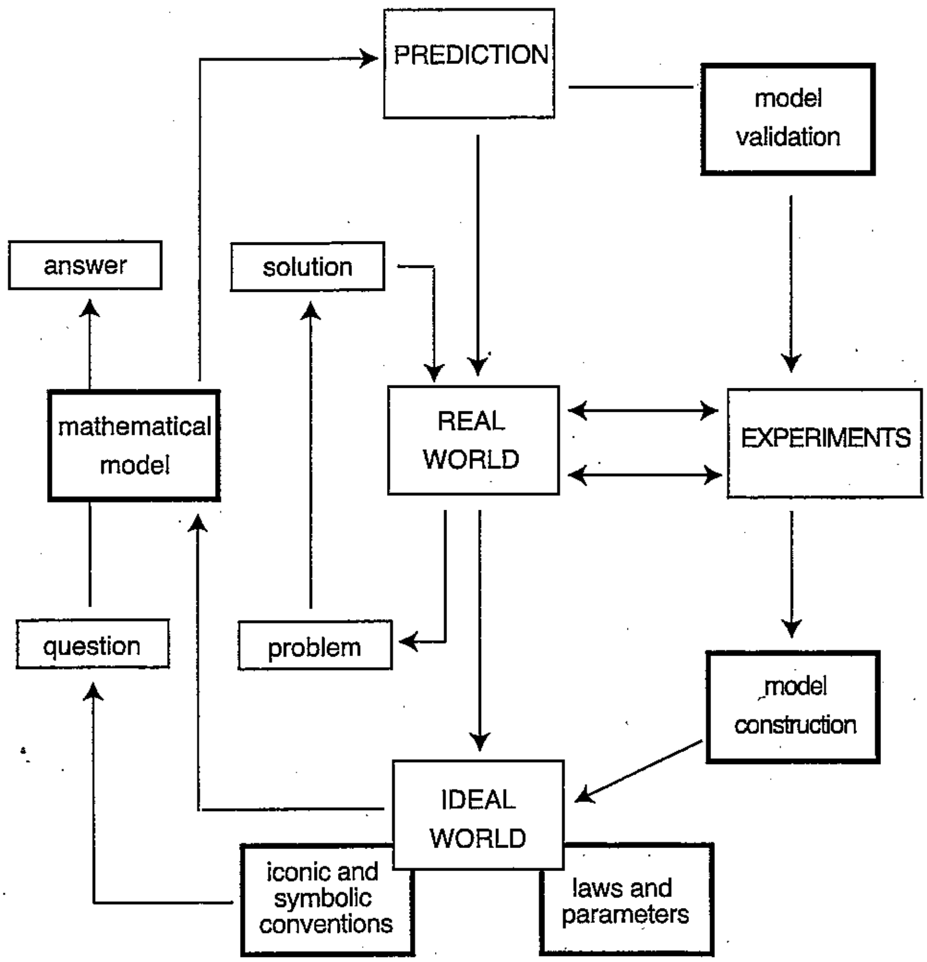
PREDICTION
model validation
answer
solution
mathematical model
REAL WORLD
EXPERIMENTS
question
problem
model construction
IDEAL WORLD
iconic and symbolic conventions
laws and parameters

Fig. 1

(B) the *real world-experiments-ideal world* loop
This is the conventional basic physics loop. Here an ideal world is generated from the real world through a strategy involving necessary simplifying assumptions together with quite elementary empirical modelling–the experiments to determine the applicable laws of physics and to fix the associated parameters relevant to the related ideal world.

(C) the *ideal world-mathematical model-prediction-real world* loop
This is the acknowledged standard mathematical modelling loop in which only the "compare with reality" box has been modified and given a dual role.

(D) the *prediction-experiments-real world* loop
This is another basic physics loop. It has been granted an enhanced significance in this scheme since without it the entire modelling activity would be rendered vacuous.

To complete the modelling diagram the traditional applied mathematics procedure has been added. Its starting point is the ideal world box incorporating the pertinent laws of physics and the parameters obtained from the associated process of model construction. The conventional iconic and symbolic regime specifically related to applied mathematics enables the model to be developed and a well-defined answer to be given to a correctly posed question. This is the traditional examination style question bearing little or no resemblance to any real world problem. It is allocated its true place in this scheme-it lies *outside* the modern mathematical modelling activity as a model apart.

3. THE INTEGRATED COURSE

As part of their two year B.Ed. programme at the University of Southampton, students took a course in mathematical modelling. The programme had to cater to their specific requirements as trainee teachers and, at the same time, had to draw upon their not too limited mathematical backgrounds. Most of the students had little or no applied mathematics background although this was compensated by the fact that they were all highly motivated mature students. Given their level of preparedness, this cohort was considered ideally suitable for an innovative approach to the teaching of elementary applied mathematics within a mathematical modelling course.

The twelve week course was timetabled for one single session (S1) and one double session (S2) each week. The students' modelling activities associated with this novel approach to applied mathematics teaching allowed one topic to be introduced each week for eight weeks. The students worked in groups of about four and, although not compelled, they were encouraged to rotate membership from one topic to the next. The remaining time was devoted to general discussions about the various methodologies encountered in other modelling courses and to the non-timetabled contact with students on a one-to-one basis in relation to their individual project work–the other component of the course. This individual project work gave students the opportunity to pursue a topic of their own choosing and this freedom invariably resulted in further work in elementary applied mathematics. The manner in which each topic was introduced to the students varied according to the specific objectives associated with the topic. The overriding principle concerned the collection of appropriate data–data for model *construction* and data for model *validation*. The mathematical expertise required of the students was kept to a minimum and, more often than not, the answer to any mathematical question was given to the students.

The assessment of the group modelling work was based almost entirely on the reports submitted by individual students. Students were expected to choose two out of the six topics designated for assessment purposes. Each account had a common element covering the group work in the two sessions devoted to the topic and for discriminatory purposes the individuality of the report was achieved through the expectation of further realistic investigations by the student in a directly related area. Thus the application of the model to novel situations was encouraged, with further mathematical models developed and/or new experimentation undertaken. The necessary literature search by the students to generate the report's distinctiveness enriched the activity and was considered by the students to be an extremely worthwhile feature of the course.

4. SOME STUDENT MODELLING ACTIVITIES

A selection of the topics employed during the short life of this innovative course will be given here, although the commentaries will lack the necessary detail for a reasonable appreciation of the implementation of the modelling scenario. The first two are presented as almost complete accounts with the others just listed for information.

Shortening of a Rope When Knotted
(Mathematical Monthly, 1989)

This is a well tested modelling exercise and is generally accepted as a suitable exercise in a variety of modelling scenarios (Open University). As far as this methodology is concerned it is ideal as the first activity at all levels since it exposes students to the fundamental aspects of the modelling processes.

S1 The facilitator had a variety of ropes of about two metres in length whose diameters ranged from 8mm to 24mm. A simple overhand knot was demonstrated and the problem posed "By how much is the rope shortened when the simple overhand knot is tied in it?" Each group of students was given a complete set of ropes and the necessary techniques for the required measurements were discussed. The accuracy of the actual measurements of both the shortenings and the diameters presented the groups with serious problems. The initial attempt to aggregate the groups' results exposed the poor experimental techniques employed since the tension applied to the ropes in the construction of the knots usually varied from rope to rope and from group to group. It was not unusual for the whole session to be repeated! Aggregation indicated to the groups that they were replicating and this instilled confidence. The linearity of the plot of the shortening against diameter was finally demonstrated.

S2 Various simple geometrical representations of the knot were developed by the groups to explain the empirical result of the previous session. Most groups got very close to a solution by systematic revisions of their model. Well satisfied, the groups then turned to the second problem "By how much is a rope shortened when a simple figure-of-eight knot is tied in it?" On this occasion the experiment was not performed first since this time the groups were expected to create a mathematical model in order to predict the shortening that will occur. The necessary refinements of the elementary geometrical representations were performed quite easily by the groups since they had already experienced a similar activity in this session. When the groups were reasonably satisfied with their model, that is, their prediction, they were allowed to perform the validation experiment. It was unusual for the students to be disappointed with their experiences.

The Pendulum–Simple and Otherwise

The oscillation of a simple pendulum is probably the most popular of all the possible demonstrations in an elementary applied mathematics course. It is used in this course not as a device for determining the value g but as a device for stimulating the students' interest in periodic motion.

S1 This session concentrated on the establishment of the empirical result that, for small oscillations, the period is independent of the mass of the pendulum bob but proportional to the square root of the pendulum's length. To this end the groups used the same selection of bobs and strings although different selections would have made the analysis more convincing. Aggregation of the groups' results was always spectacular. The mathematical model associated with the empirical result was then given to the students–the application of Newton's second law to obtain the equation governing simple harmonic motion.

S2 Before this session students were expected to search the literature for the large angle theory of the simple pendulum. Thus armed with a variety of textbook results it was possible for the standard mathematical model required in this session to be reviewed for them. The prediction having been agreed, the validation experiment was performed. Good results were obtained if the effect of air resistance was conveniently acknowledged.

Some of the other topics

(i) Torricelli's law (Orman, to appear)

(ii) Hooke's law, normal modes

(iii) Newton's experimental law of restitution

(iv) Archimedes' principle, buoyancy and stability

(v) Chains and Strings

(vi) Moment of inertia, flywheels

5. CONCLUSIONS

Since the students did not progress to further applied mathematics courses it was difficult to judge their level of understanding of the basic principles. However the quality of the further work undertaken by the students as part of their individual submissions indicated an appreciation of the subject matter not often encountered in the work of students following a conventional course.

The students were enthusiastic and highly motivated during the course and many stated that, although they were quite apprehensive about,the subject before the course, this partial insight had given them confidence to read and comprehend applied mathematics textbooks. They all agreed that the course had given them an awareness of some parts of applied mathematics that would help them become better teachers of mathematics. Indeed, as Alexander Pope put it,

"Let such teach others, who themselves excel
And censure freely who have written well

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