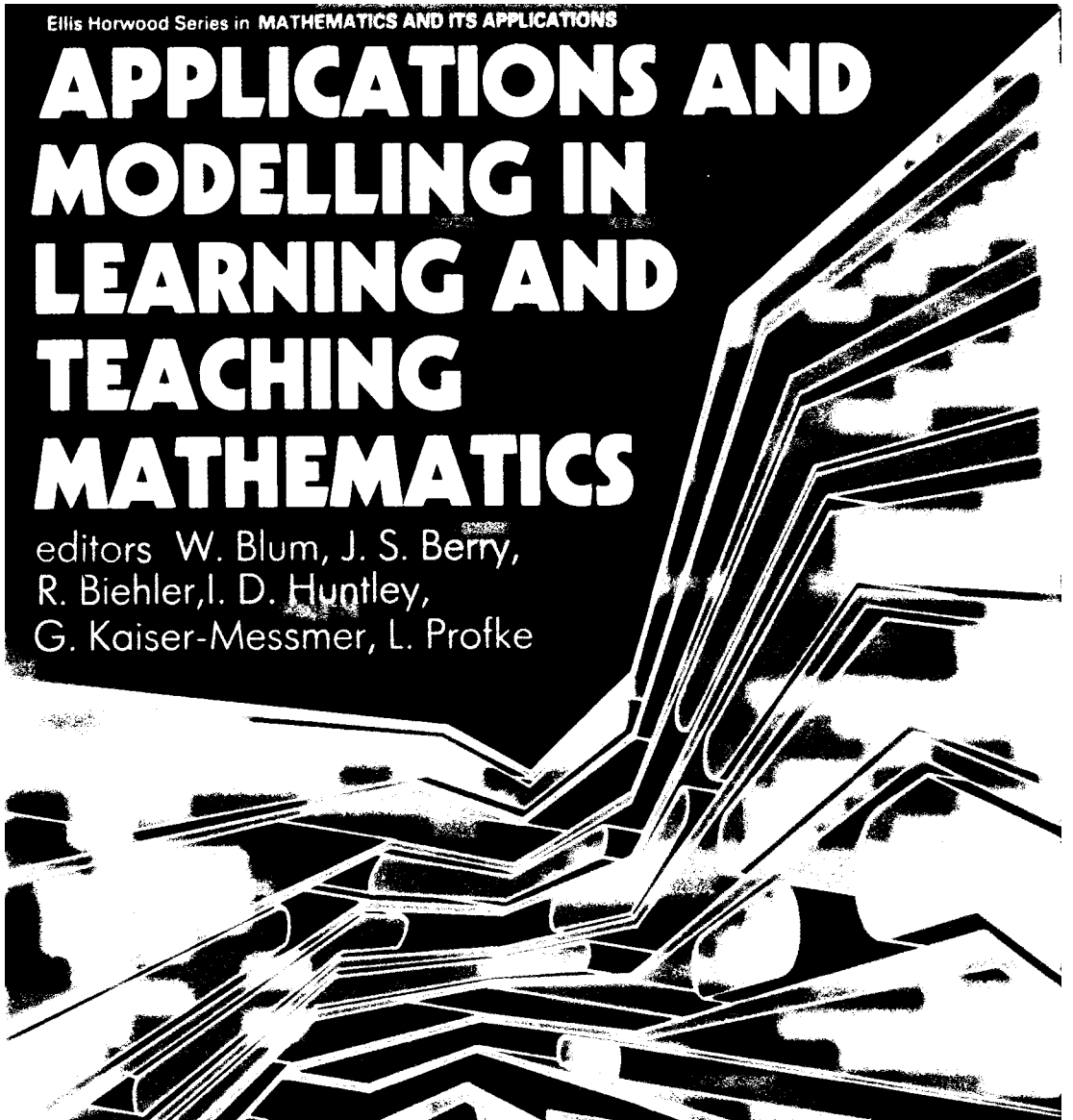
Ellis Horwood Series in MATHEMATICS AND ITS APPLICATIONS

APPLICATIONS AND MODELLING IN LEARNING AND TEACHING MATHEMATICS

editors W. Blum, J. S. Berry,
R. Biehler, I. D. Huntley,
G. Kaiser-Messmer, L. Profke



Ellis Horwood Series in
Mathematics and its Applications

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APPLICATIONS AND MODELLING IN LEARNING AND TEACHING MATHEMATICS

Editors: W. BLUM, Department of Mathematics, University of Kassel, Federal Republic of Germany; J. S. BERRY, Department of Mathematics and Statistics, Plymouth Polytechnic; R. BIEHLER, IDM, University of Bielefeld, Federal Republic of Germany; I. D. HUNTLEY, Head of Department of Mathematics, Sheffield City Polytechnic; G. KAISER-MESSMER, Department of Mathematics, University of Kassel, Federal Republic of Germany; and L. PROFKE, Department of Mathematics, University of Giessen, Federal Republic of Germany

This book presents the appropriate recent research and development studies in the teaching and learning of mathematics in connection with real-life situations, e.g. theoretical concepts, empirical research, concrete applications, and classroom examples for all levels from lower secondary up to university. It provides an international forum for the discussion and exchange *of experiences with applications and* modelling, considering in particular the social and technological developments and changes with respect to mathematics teaching.

The book commences with various theoretical aspects of applications and modelling in the learning and teaching of mathematics. Chapters 1 to 5 contain the main lectures given at the Third *International Conference on the* Teaching of Mathematical Modelling and Applications (1987) by five internationally-renowned experts in the field of mathematics education: H. Burkhardt (UK), R. Fischer (Austria), M. Niss (Denmark), H. Pollak (USA) and H. Schupp (Federal Republic of Germany).

Section A concentrates on various theoretical aspects of applications and modelling in the learning and teaching of mathematics: philosophical and epistemological aspects, empirical investigations, methodological questions and the problem of assessment. Section B contains examples, materials, ideas and projects related to applications and modelling in mathematics instruction at the lower secondary school level. In particular, many tried and tested teaching units for ages 11–16 are presented. Section C contains examples, materials, ideas and projects related to applications and modelling in mathematics teaching at the upper secondary school level, including vocational education. Again, most of the chapters present concrete examples for ages 16–19. The chapters of Section D deal with applications and modelling in mathematics teaching at universities and polytechnics. Special reference is given to modelling courses. Several examples of such courses from different tertiary institutions are presented, including some for initial teacher training. Furthermore, examples and analysis of applications are given which are suitable for mathematics lectures at tertiary level.

Microcomputers are gaining more and more importance in our society, and also in the educational system. Section E refers to various aspects of the use of micros in mathematics teaching at school and university. Several chapters describe possibilities for the use of computers as a tool in mathematics teaching, others for simulation purposes. Some contributions report on empirical research concerning the use of microcomputers in the classroom, and others relate to suitable software or to computer graphics.

Readership: Mathematics, pure and applied; Mathematical modelling.

APPLICATIONS AND MODELLING IN LEARNING AND TEACHING MATHEMATICS

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Mathematics and its applications are now awe-inspiring in their scope, variety and depth. Not only is there rapid growth in pure mathematics and its applications to the traditional fields of the physical sciences, engineering and statistics, but new fields of application are emerging in biology, ecology and social organization. The user of mathematics must assimilate subtle new techniques and also learn to handle the great power of the computer efficiently and economically.

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APPLICATIONS AND MODELLING IN LEARNING AND TEACHING MATHEMATICS

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Preface

This book contains the five main lectures as well as 65 selected papers presented at the *Third International Conference on the Teaching of Mathematical Modelling and Applications (ICTMA-3)*, Kassel University (FRG), 8–11 September 1987. This conference was the third in a series of biennial international conferences on the teaching of mathematical modelling and applications.

THE HISTORY OF ICTMA-3

Relationships between mathematics and the real world, *applications* of mathematics, have existed as long as mathematics itself. During the last few decades an enormous extension of the use of mathematics in other disciplines has occurred, and many new sciences have been 'mathematised'. Nowadays the concept of a mathematical model is often used and, especially in the past two decades, interest has turned to the dynamic interaction between the real world and mathematics, to the process of translating a real situation into a mathematical model and vice versa, that is to *mathematical modelling*.

The continued growing importance of mathematics in everyday practice has not been reflected to the same extent in the *teaching and learning* of mathematics in school and university. In particular the world-wide New Maths movement of the nineteen-sixties actually caused a reduction of the importance of applications in mathematics teaching – contrary to the original intentions of the reformers (which were detailed in the proceedings of the famous conference at Utrecht in 1967). Eventually, in the nineteen-seventies, there was a reaction to the excessive formalism of New Maths, and a return in many countries to the importance of applications and connections to reality in mathematics teaching. However, the main emphasis was put on models.

Then, especially in Great Britain, it was pointed out that the dynamic interaction between the real world and mathematics, which is importance in practice, should also be taken into consideration in classes at school and university. Thus, for the first time, the relevance of *teaching mathematical modelling* was emphasised, for example by the Spode Group, the Mathematics Applicable Group, the Shell Centre, the Open University and by several groups at polytechnics and universities.

The return to applications in the nineteen-seventies has also become apparent in national and international *conferences* on mathematics education where increasing stress was put on the topic of applications, for example at the Third International Congress on Mathematics Education (ICME-3), Karlsruhe 1976, or at ICME-4, Berkeley 1980. There were also some special congresses on applications at school level: Echternach 1973 (in the tradition of Utrecht 1967), Klagenfurt 1976 or Bielefeld 1978. Then, at the beginning of the nineteen-eighties, the time was ripe for a particular series of conferences on teaching applications. Owing to the activities in Great Britain the first such conference was held at Exeter University in 1983†, with the emphasis on modelling.

One last point in this short review. The topic of 'the relationship to reality in the teaching and learning of mathematics' is relevant here and now all over the world. For, everywhere in the world, pupils and students are to be educated to be responsible citizens, are to be trained towards professionalism at work, and are to acquire appropriate intellectual attitudes. In this mathematics plays an essential part, especially mathematics in connection with everyday life, with the world around us, and with other subjects and disciplines. For – to quote Henry Pollak in his opening lecture in Exeter in 1983 – 'Society provides the time to teach mathematics in our schools every year. Why? Not because mathematics is beautiful – which it is – or because it provides great training for the mind, but because it is so useful.' The problems in learning and teaching arising here are often identical in different countries. Therefore it makes sense to communicate with one another, to exchange experiences and ideas, and to learn from one another. In addition, there are sociocultural differences or, rather, different accentuations in the pedagogical discussion, which can only be understood in the light of history, and one can learn from these differences. Thus it is useful and extremely profitable to organise such periodical conferences on an *international* basis.

The third conference of this kind was held at Kassel University in 1987. It was held there because in Germany there is a strong tradition in the field of application-oriented mathematics instruction. Compared with the first and second conferences, there was a deliberate *extension* of the range of topics and aspects. In the Kassel conference programme

- pedagogical and didactical aspects as well as basic theoretical questions were extended and made more clearly recognisable
- the complete range of ages from lower secondary up to university level was considered
- tried and tested teaching units and ideas were emphasised more intensively
- questions concerning computers were given a more prominent place.

Certainly this conference brought further progress with regard to these and other questions, and probably we are – as David Burghes said+ at the second conference in 1985 – 'heading in the right direction' with respect to mathematical modelling and applications.

† See conference proceedings *Teaching and Applying Mathematical Modelling*, edited by J. Berry *et al.*, Ellis Horwood, Chichester, 1985.

+ See conference proceedings *Mathematical Modelling Methodology, Models and Micros* and *Mathematical Modelling Courses*, edited by J. Berry *et al.*, Ellis Horwood, Chichester, 1986 and 1987.

We hope that this progress becomes apparent in this book. We also hope that the Kassel conference, as well as its proceedings, helped to consolidate the tradition of biennial international conferences on the teaching of mathematical modelling and applications†.

THE STRUCTURE OF THIS BOOK

All the contributions in this book deal, in one way or another, with applications and modelling in the learning and teaching of mathematics at school or university.

Chapters 1–5 contain the *main lectures* given at the Kassel conference by five internationally renowned experts in the field of mathematics education: H. Burkhardt (UK), R. Fischer (Austria), M. Niss (Denmark), H. Pollak (USA) and H. Schupp (FRG). The following chapters are divided up into *five sections*. Of course, this division is by no means intended as a hard and fast classification of the contributions. In fact, most of the chapters could fit in more than one section, but have been allocated to a particular section in order to aid the reader.

The contributions of *Section A* concentrate on various *theoretical* aspects of applications and modelling in the learning and teaching of mathematics: *philosophical* and *epistemological* aspects, *empirical* investigations, *methodological* questions and the problem of *assessment*.

Section B contains *examples, materials, ideas* and *projects* related to applications and *modelling in mathematics instruction at the lower secondary school* level. In particular, many tried and tested teaching units for ages 11–16 are presented.

Section C contains *examples, materials, ideas* and *projects* related to applications and modelling in mathematics teaching at the *upper secondary school* level, including vocational education. Again, most of the chapters present concrete examples for ages 16–19.

The chapters of *Section D* deal with applications and modelling in mathematics teaching at *universities* and *polytechnics*. Special reference is given to *modelling courses*. Several examples of such courses from different tertiary institutions are presented, including some for initial *teacher training*. Furthermore, *examples* and *analyses* of applications are given which are suitable for mathematics lectures at the tertiary level.

Microcomputers are gaining more and more importance in our society, and also in the educational system. *Section E* refers to various aspects of the use of micros in mathematics teaching at school and university. Several chapters describe possibilities for the use of computers as a *tool* in mathematics teaching, or for *simulation* purposes. Some contributions report on *empirical* research concerning the use of microcomputers in the classroom, and others relate to suitable *software* or to *computer graphics*.

† *The fourth conference of this series (ICTMA-4) will be held at Roskilde University (Denmark), 3–7 July 1989.*

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1

Mathematical Modelling in the Curriculum

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1. A FRAMEWORK

In this chapter I shall look at the development of the teaching of mathematical modelling over the last quarter-century or so, partly in celebration but mainly to suggest some ways in which further progress might be achieved. I shall begin with a little history: though I am no scholar in this domain, a few mileposts (or their metric equivalents) are helpful in establishing a perspective of what has been achieved.

The period 1955–60 produced the drive towards 'modern mathematics', a fundamental revision of the mathematics curriculum in many countries. The emphasis in the changes sought varied greatly. In England there was a strong emphasis on applications, on the importance of using mathematics, that sprung from a long and distinguished historical tradition from Newton through Maxwell to Dirac and beyond. In the outcome, some ten years later, the text books which dominated in English schools showed hardly a trace of any applications of mathematics: there is a cautionary lesson here on the dynamics of curriculum change to which I will return in Section 3.

However, this wave of reform triggered the development of the teaching of mathematical modelling in the UK in the 1960s. It was thought a good idea that applied mathematics should be looked at, and I became involved. Applied mathematics in the UK, taught in the age range 16–18, was then the Newtonian mechanics of particles and rigid bodies. By the end of a year-long course for teachers, I had moved to a view that the situation in applied mathematics was actually a great deal more serious than in pure mathematics – the standardised and stylised nature of the problems (ladders against walls, projectiles on planes, colliding balls, and so on) and the absence of any serious, flexible discussion of the link between the real-world situation and mathematics seemed inexcusable. Thus for me in 1962 began conscious work on the teaching of mathematical modelling.

Over the next few years I and my colleagues devised learning activities, mainly with undergraduates to whom we had ready access, but also with school teachers and their students. The approach was mainly informal, getting students to tackle unfamiliar practical problems with encouragement but with a minimum of intervention. We soon started looking at the problem solving processes and the strategies involved, and familiar diagrams began to emerge (Burkhardt, 1981; Treilibs *et al.*, 1980). Among others, George Hall's contribution should be mentioned for its range and its creative imagination. One example must suffice – an undergraduate course entitled Information Theory, which has proved a fertile domain for stimulating students to work of high quality over an enormously broad range of practical problems, from the loading of roofing tiles onto a truck to the dating of English churches.

It is no accident that this curriculum innovation began in the universities: it is a feature of universities that, because of the great independence of individual lecturers in both course design and assessment, *it is easy to innovate. Unfortunately, it is* correspondingly difficult to institutionalise an innovation so that it persists when the lecturer concerned moves on to other responsibilities or other places. Equally, it is easier to innovate in elementary schools than in secondary schools for various reasons – the teachers are more flexible and, covering the whole curriculum, less bound by rigid and traditional subject definitions. The timetable is also more flexible.

The next crucial initiative was in the USA, in elementary schools. The USMES project (Unified Sciences and Mathematics in Elementary Schools) was inspired by, among others, Henry Pollak, led by Earle Lomon and funded by the National Science Foundation in the US in the late 1960s. It introduced real problem solving to the elementary school classroom on the basis of substantial six-week projects carried through on a whole class basis. Each project was based on a challenge which had been shown to lead to rich and tractable classroom activities and to decisive conclusions, with a strong emphasis on action outcomes. It remains an outstanding achievement. Its weakness was in the high demands it made upon the teachers involved, both in terms of commitment and the range of skills that were necessary. The great majority of teachers are accustomed to working with detailed guidance and support, even in familiar areas of the curriculum. I do not believe that the level of support offered would have been adequate for most teachers.

The 1970s saw the establishment of some other islands of activity around the world. Good books were written with a modelling emphasis but, as so often, they were not widely used. In the UK, the Spode Group was established: it brought a breath of fresh air by introducing a wide range of applications of mathematics in the form of suggested lessons. Many new models were introduced, though often the amount of actual modelling by the students seemed to be relatively small – the situations were on the whole too advanced to be within their reach, so models were provided. The evidence on the classroom outcomes in typical classrooms, or indeed on the extent of use of such work, is unclear. Indeed, there was no climate to encourage such changes, so one would expect take-up to be confined to the small proportion of innovative teachers.

The situation in the UK is now changing, largely through changes in the examination system flowing from the *Cockcroft Report* (1982), and it will be interesting to see what happens in practice. One clear success can be noted – an alternative examination syllabus in *Decision Mathematics* (Burghes *et al.*, 1981), with an emphasis on operational research, which allows schools to do such work as an alternative to mechanics in the 16–18 age

range: here there has been a significant take-up. Though the modelling element is relatively small, it represents an important extension of the range of applications of mathematics available to schools.

However, the main success in the development of modelling as a curriculum component in the UK has been in undergraduate courses in polytechnics. These institutions have been created over the last 20 years, with university status but a more practical emphasis. Their degree-giving status has until recently been vested in a national supervisory body, the CNAA (Council for National Academic Awards), which monitors and approves all such courses. The 'benefits of colonialism' in such a situation have been dramatic – once a case for modelling as an integral part of the mathematics curriculum was established with this body, all such courses were effectively required to include it. This institutionalisation is reflected in the high proportion of participants from *British polytechnics* in this and previous conferences in this series. The contrast with *British universities* is stark: the distributed curriculum authority which makes experiments so easy means that institutionalisation is slow, only arising when there is a universal consensus that change cannot reasonably be avoided. I think that all of us here who come from British universities are aware that the modelling courses we have developed are likely to die if we ourselves stop teaching them. The polytechnic situation parallels that in schools, though the system-inertia to change is inevitably greater there.

Thus at the present time modelling is a substantial reality in UK polytechnics. It can also be discerned in nearly every classroom in the early years of schooling from age 5 to 7, in the UK as elsewhere in the world. Elsewhere in the curriculum, it is still a matter of hopeful signs rather than substantial presence. In my talk at the previous Exeter conference (Burkhardt, 1986) I discussed the problems of providing adequate support for teachers, of a kind that would enable them to face such changes and implement them effectively and with reasonably comfort and enjoyment to themselves and their pupils. In later sections I shall describe what we have been doing in this regard and the problems that remain.

We have been operating in a national climate that is in principle highly favourable for changes of this kind. In 1982 the *Cockcroft Report* set out in some detail what should be in the mathematics curriculum, and described some elements that would be needed to achieve the changes involved. It was welcomed very widely indeed and accepted by the Government as the basis for its future planning. It gave, as usual in England, strong emphasis to applications and in its most quoted paragraph (243) demanded a pattern of learning activities that implies a modelling approach. This 'balanced diet' of learning activities involves six elements:

- exposition by the teacher;
- discussion between teacher and pupils and between pupils themselves;
- *appropriate practical work*;
- consolidation and practice of fundamental skills and routines;
- problem solving, *including the application of mathematics to everyday situations*;
- *investigational work*.

At present the overwhelming majority of classrooms score 2 out of 6 on this list of requirements: there are no prizes for identifying which two, since this emphasis on imitation by the students seems to be true worldwide. The problems of creating a

substantial curriculum component based on **autonomous student activities** remains central to our interests, since student independence and responsibility are the core of modelling – the integration by the students under their own command, of the various elements in Fig. 1.

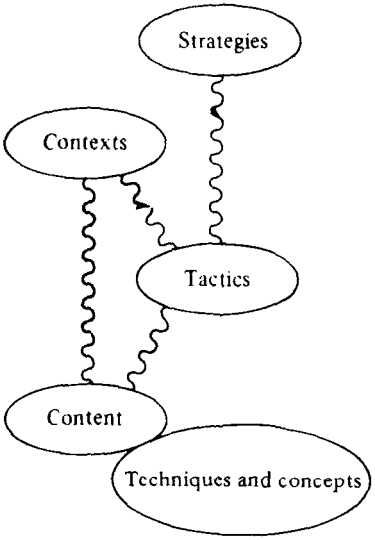

Fig. 1 –

Finally in this introduction a little classification and definition of terms. The distinctions are crucial though, as with any taxonomy, different words could be used. I shall be brief because many of you are familiar with the issues involved. Like some of the other elements of this introduction, they are discussed more fully in my book *The Real World and Mathematics* (Burkhardt, 1981).

Applications and **modelling** are both the concern of this conference: the latter is that subset of the former in which the task presented to the solver is non-routine, in some way unfamiliar, and the finding or designing of an appropriate model is part of the task. Many applications are quite rightly presented to the student essentially complete, as important things to know, or simply as impressive achievements of the human mind. Here two differences of approach are worth recognising – an application may be presented as an **illustration** of the power of a particular mathematical technique: this has value in a number of ways, including providing a concrete illustration of the mathematical concepts involved and providing **standard models** for use by the student, perhaps with some adaption, in that field of application. Alternatively, the focus of an application may be on the practical situation that is being studied, and how it may be better understood through the use of mathematical models. In my view it is essential that the students know whether they are being asked to operate in an **illustrative** or a **situational** way, because the appropriate responses are different: in the former the aim is to show off as much mathematics as you can, while in the latter power over the situation is the key – power

of understanding, of decision and of advocacy. Using more mathematics with no extra payoff is actually counterproductive.

A third important distinction is between **standard models** and **new situations**: for the latter we need to think of the question as to 'how new' they are and the ability of students to **transfer** knowledge to new situations at different 'transfer distances'.

Finally a comment on **interest levels**. I have found it useful to distinguish Action problems, Believable problems, Curious problems, Dubious problems and Educational problems. From USMES onwards it has been widely recognised that Action problems (those that affect your own life and work) have advantages from a curriculum point of view in terms of motivation, commitment, understanding of the practical situation, and the ability to evaluate the model. In our work on *Numeracy through Problem Solving* (Binns *et al.*, 1988), we have discovered that the best Action problems seem to involve a *mixture* of practicality and fantasy – this result surprised me a little at first. It has an important educational spin-off. In general, weak problem solvers tend to solve just one problem, while those who are more effective solve a **'space'** of problems in which a particular solution is but one point or sub-space in many dimensions. This kind of generalisation is of the essence of fantasy.

Though all these things are included within the field of applications, and are important to the curriculum, modelling occupies a special place. Modelling is not about illustrations or standard models – it is about new situations in which transfer is involved. Curious problems are important but not enough: Action problems should also be involved. However I warm to the opposite suggestion, that it is a mark of superior performance as a mathematician to be able to tackle effectively problems in which you are not at all interested! However, I am sure that the mathematics curriculum has had far too many of these in the past: one of the real achievements of the modelling movement has been to reduce their dominance by introducing more Action, Believable as well as Curious problems.

Finally, we have **'the big hole'**. Mathematics in schools and universities is dominated by the learning of manipulative techniques. All the techniques taught at school level can now be executed more quickly and accurately on cheap technological devices – this applies not only to arithmetic (a calculator) but to algebraic manipulation, statistical manipulation, graphical manipulation and so on. (It is true that at present it is harder to program the algebraic manipulator than to do school level algebra, but that will soon change.) It is not yet known how much manipulation is needed in order to consolidate the acquisition of the mathematical concepts involved, but it is fairly clear that this will be far less than the amount now included in an often ineffective effort to produce fluency and accuracy with pencil and paper. Thus the big hole. Modelling and applications are obviously strong candidates for a part in filling it, but only if they can be developed to be accessible to the students and, particularly, to the teachers who will be involved. The people at this conference will inevitably be at the core of that effort. The time to deliver is soon.

2. DEVELOPMENT METHODS FOR TEACHER SUPPORT

It is clear that the introduction of applications of mathematics, and of modelling in particular, into the mathematics curriculum presents most teachers with a formidable challenge. It requires a much broader range of teaching strategies than the standard explanation + example + exercises approach to mathematics teaching. If change is to be

brought about on a large scale, an appropriate combination of **pressure-for-change** and **support** has to be developed. In England the public examination system exerts a very powerful effect on the secondary mathematics curriculum, in that the kind of tasks included in the examinations predominate in the curriculum. Indeed, all successful innovative development projects have included an alternative examination matching the project: however, the proportion of schools using such examinations is usually quite small, so it seemed worth trying an alternative approach. In this section I shall describe our work with the largest examination board in England, the Joint Matriculation Board (JMB), on the promotion of **well-supported gradual change.**

For the last five years the JMB and the Shell Centre have been working together to explore and develop a new modular approach to syllabus change. The first module of this Testing Strategic Skills (TSS) programme was introduced into schools in preparation for the Board's 1986 examination in mathematics for 16-year-old students. This first module *Problems with Patterns and Numbers* (1984) was concerned with problem solving within pure mathematics itself: the extra demands here are a step on the way to teaching mathematical modelling, which is the concern of the third and subsequent modules which are described in other papers in these Proceedings (Binns *et al.*, 1988). Here we outline this approach to supporting curriculum change.

The **key principle** of the TSS programme is a cautious realistic approach, based on empirical development in the classroom, to introducing new examination tasks that will encourage curriculum changes that are widely recognised as desirable.

The **essential elements** in implementing this approach are the following.

(1) ***Gradual change.*** We thought in terms of changing the basis of just one question in any year, corresponding to about 5% of the syllabus or three weeks' teaching. Why this modular approach? Everybody, including teachers, finds it difficult to substantially change what they do and this is particularly true when rather different methods of working are involved. Thus sweeping syllabus reforms have often had limited success, either because they have been taken up only by a small minority, or because they have been confined to details of subject content – the easiest changes to absorb.

(2) **Effective support.** Teachers are very busy people and most base their teaching on well-defined schemes of work, even in areas with which they are familiar. In asking them to move in to less familiar topic areas or modes of classroom learning activity, it is important to offer them something equally supportive – the module teaching package is developed to do this. Of course, teachers will do things in their own way in their own classroom, and there will always be a range of approaches and materials for teachers to choose from, or to develop themselves. The module package is offered as one proven possibility.

(3) **Empirical classroom development.** It is not possible to guess reliably what the effects, in the classroom or in the examination, of any significant change will be. As a result, the outcomes are often quite different from the original intention in important respects. In the TSS programme these dangers are much reduced by an empirical approach in which all the elements of the module are developed and tested until they work in the way intended with a broad sample of teachers and children, representative of those who take the examinations concerned. It is here that the Shell Centre plays the central role: it has developed new research and

development methods for this task, particularly emphasising structured classroom observation of teachers and pupils in action (Beeby, Burkhardt & Fraser, 1980). These are crucial for the refinement of the materials, so that they work in a robust and enjoyable way for teachers with a wide range of styles and approaches.

Each Module Package consists of three components:

(1) **Specimen examination questions**, with marking schemes and sample (not 'model') pupils' answers.
(2) **Sample classroom materials** to support about three weeks' teaching which prepare children for the new types of question. The materials include pupil worksheets, notes for the teacher, and extracts from pupils' work.
(3) **Support materials** that help teachers adapt to new aspects of classroom activity or teaching style – and to extend them to other appropriate parts of their teaching with other classes, where equally supportive classroom materials **are not available**. This component can be viewed as a do-it-yourself in-service course.

The second module is on *The Language of Functions and Graphs* (1986). It aims to develop children's fluency in 'speaking mathematics' – and in *translating* information about the real world from one form to another. These are important modelling skills which are seriously lacking, even in many able pupils who can *manipulate* mathematics fluently (Fig. 2 gives a typical example).

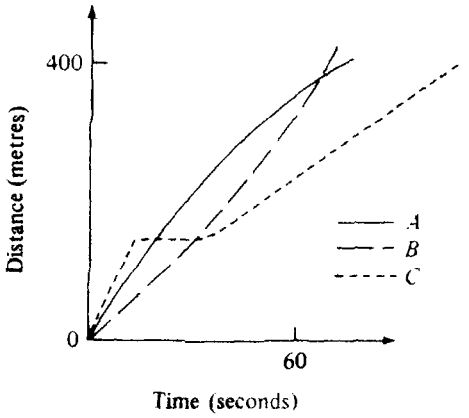

Fig. 2 Describes what happened in this 400 *m* hurdles race.

Our application of this approach to the teaching of mathematical modelling has been in a project called *Numeracy through Problem Solving*, again in cooperation with the Joint Matriculation Board. The key aspects of teacher support and of assessment are described in other papers in this volume (Binns *et al.*, 1988).

3. THE DYNAMICS OF CURRICULUM CHANGE

This brings me back to the strategic issues involved in bringing modelling into the curriculum at all levels. **Developing curriculum development** is, I believe, an important new area for study. To establish this, I shall mention four key aspects, though I can only do so briefly here. Is there an established **need**? What are the **opportunities** for progress? Are there some examples of work that obviously needs to be done? What further areas for study present themselves?

The **need** is, to me at least, self-evident. There is no established method of planned curriculum change, at least in many countries including the UK. Many serious curriculum initiatives have been undertaken, and even widely agreed, and yet observation of typical classrooms five or ten years later shows a qualitative mismatch between the intentions and the outcome. So planned curriculum change is very difficult.

What are the kinds of failure? **Low take-up** is common, with most classrooms showing no sign of the innovation. **Dilution** describes the transformation of the intention (often brilliantly realised in the classrooms of the exceptional teachers used in the development process) into a pale shadow of itself in the typical classroom. Finally there is **corruption**, where some superficial aspects of the innovation are taken on board but its principal aims are ignored: this is particularly severe when changing roles and responsibilities in the classroom are involved – in routine problem solving the teacher hurries to explain or direct, in real problem solving it is made clear to the students that the practice of mathematical technique is actually the teachers main aim, and so on.

In advancing any innovation, there is a 'catch', a dilemma that is particularly acute for publishers or those for whom financial constraints are dominant, but which applies to us **all**:

> If an innovation is not **close** to the familiar, few people will buy it; if *it is* close, why bother to develop it.

It is this dilemma that has led us to develop an approach based on gradual qualitative change, where teachers are asked to change only a small part of what they teach each year (a few weeks with one or two classes) but to change it profoundly and qualitatively. Most of what they do is left unchanged and unthreatened – for this year! Provided support is there which gives teachers confidence that they can implement the innovation reasonably faithfully, we have found no resistance to such change. One should however constantly ask oneself

Why should *they* change?

where 'they' may be anyone in the system, but particularly the teacher who faces the greatest challenges.

It is a remarkable feature of education that *so little* research and development goes into finding better methods of curriculum development and research. This is a major aspect of activity in many fields in which major progress is rapid, but hardly occurs in education.

What are the opportunities? To make some progress with this a framework is needed.

We adopt a systems approach of a kind familiar in engineering, say, but not in education. education. The system is

teachers teaching children – in classrooms – in schools – in society.

Our aim is to change the way the system operates in various ways. There are various levers or stimuli at our disposal, but we cannot be sure what the effects will be. An empirical approach is therefore appropriate; we think it needs to be both practical and pragmatic, fundamental and profound. In order to make the system work better we need to study its operation at all levels, but also to learn more about its underlying dynamics. I call this The Bell Model: they run a telephone system and, to advance that end, find it worthwhile to conduct fundamental research at the highest level in many areas from solid state physics through to information theory, which was invented by them. Of course, the education system is very different in many ways and analogies are always imperfect, but they can be informative. A comparison with medicine a century ago is also useful – fundamentally a craft profession in which more or less effective rules of behaviour were passed on from one generation of practitioners to another, the defects of this approach became evident (leeching may not cure everything) and it gradually but slowly became accepted that a research based approach had something to offer. What are the analogous aspects in education?

I find it useful to identify various levels of research and development. These levels are:

L – **Learning research** is concerned with the individual students, their understanding and their misconceptions and how these developed (10^1 children minimum).

T1 – **Teaching approaches** are developed and researched, seeking stimuli for learning that will be more effective (10^2 children).

T2 – **Realisable teaching** is focussed on what can be achieved in practice by typical teachers in realistic circumstances of work and support (10^3 children).

C – **Curriculum** change on a large scale brings in as well as the student and teacher variables those of the school, the school district and society at large (10^4–10^7 children).

Much more work has been done at L and T1 levels than at T2 and C level, partly for practical reasons but mainly because of a lack of attention to systems issues. The research interest is focussed on the child, almost to the exclusion of all else. This is not the only domain in which there are difficulties. Indeed in our work almost all the difficulties involved in the realisation of curriculum change arises from the difficulties of teachers, and the inadequate support provided to them by the system. Owing to this earlier neglect of so many important aspects and variables, the opportunities to do better are wide open.

There are many areas in which there is room for more imaginative and more systematic work on finding better methods of research and development, but I have space to mention only one. However, it is an area of crucial importance – that of feedback.

Feedback is a crucial aspect of any system, in particular of its stability and controlability. Even with something as well understood as sending a rocket to the moon, where the laws are known precisely in advance, the system must include essential mechanisms for detecting deviations from the intended path and for applying mid-course

corrections. In the education system, where the underlying dynamics is in many respects unknown, appropriate feedback mechanisms are essential. We have a working principle

'fail fast – fail often'

whose essential message is that the more rapidly you can learn of the defects and strengths of your ideas in practice, and revise them, the better the resultant outcomes can be.

Feedback must be adequate in **dimensionality** – that is, in the space of variables it covers. In driving a car, you need to know not only the speed of the car but also the position of the steering wheel to understand where the car is going. In education you need feedback on all the *essential* variables for your purposes; in practice this means being highly selective but not too restrictive. Different timescales must be properly balanced – the observation time of feedback must be shorter than the time over which interesting things occur, while the response time in which action is taken upon the basis of the evidence from feedback must be short compared to the project time, if several cycles of improvement are to be possible. Both of these principles are often ignored. How best they can be used in detail is a matter for experiment. The quality, quantity and cost of feedback must be borne in mind.

For example, curriculum materials that produce excellent results in a few classrooms are often, perhaps usually, not widely adopted. What feedback has been used in their development? Two kinds are common – testing by members of the author team, and written reports by trial teachers. Author testing frequently leads to some revisions but has severe limitations; the members of the author team have much more information on the intentions behind and the ideas for use of the material than are actually written down, because they have taken part in all the formative discussions. Reports by trial teachers on the other hand tend to be very sparse, providing little detailed information. What are the dangers? When only exceptional teachers are included in the trials, materials are not usable in practice by more typical teachers. When the trial sample is fully representative the materials are far from being optimised and are thus not as ambitious as they could be. How might this be improved?

We have developed the use of structured classroom observation of trial lessons as a source of feedback, and have found it invaluable in developing materials which lead to lessons that please the teachers, the students and ourselves as designers. The amount of rich detail that an observer can bring back is immense, and it can be closely related to the key design intentions and questions. The data typically are of three kinds:

(1) some simple information on the lesson, and the teachers response to the materials;
(2) an open-eyed view by the observer of significant things that happened, of short interviews with students during the lesson, and of with the teach afterwards;
(3) a structured qualitative record of the rhythm of the lesson itself.

The first two elements are familiar; the last is the core of this development method (Beeby *et al.*, 1980; Burkhardt *et al.*, 1983). This approach was developed in the context of the design of program teaching materials, based around a single microcomputer in the classroom, through the ITMA collaboration (Investigations on Teaching with Microcomputers as an Aid). Computer assisted learning materials are expensive to develop, so this approach to making sure that the materials worked really well seemed justified in

that context. However, the gains were such that we have since applied a similar approach to the development of all materials which aim to combine ambitious and challenging changes in classroom learning activities with accessibility to the typical teacher. Perhaps the most difficult of these challenges so far has been presented by the material on the *Numeracy through Problem Solving*. The gain in the effectiveness of the materials, in both positive aspects and the evidence of negative experiences from the teacher, is qualitative. We see this in the changing classroom reactions as each successive revised version is trialled.

Some other priority areas in which such work is needed include in-service training, and the management of curriculum development teams. It is astonishing to me that in England it is rare for in-service training procedures to be evaluated in terms of changes in the teachers' behaviour in the classroom – surely the prime aim. Again, traditional evaluation of the methods are not good enough, but it is a tractable problem. Equally surprising perhaps, there is no profession of curriculum designer; major projects are normally established with teams of experienced and enthusiastic *practitioners* but including no-one who has ever directed such a development project before. It is as though television programmes were made without a producer or director, simply by offering the use of the technical facilities of the studio to those with a message to convey – politicians or educationalists. It would be good to explore what benefits might occur from a team approach in curriculum development, parallel to that in television.

My excuse for spending time on methodology is that, in modelling terms, these are probably the key factors in determining whether, and how effectively, we can introduce modelling into the mathematics curriculum.

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2

Social Change and Mathematics

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COMPLEXITY, SELF-ORGANISATION AND SELF-REFLECTION

Before speaking about the potentials of mathematics I want to indicate how I see the situation of human society today. I am not a social scientist and therefore by diagnosis will be superficial and, moreover, it must be based on the views of experts. The American social research John Naisbitt (1984) has pointed out ten so-called 'Megatrends', which he assumes will play an important role in the future of at least the western hemisphere. Among others the following trends were discovered:

- the necessity for strategic thinking in terms of big systems (temporal, spatial, etc.) considering networks of ecological, economic, technological, social, . . . kinds;
- a trend towards de-centralisation, from hierarchy to social networks;
- a trend from either/or thinking towards a diversity of roles.

In a certain sense these trends seem to be a response to a phenomenon which is also frequently discussed today: *complexity*. Recently I had the opportunity to participate in a conference about the *Complexities of the Human Environment* (organised by the Honda-Foundation in July 1987 in Vienna). Many contributions to that conference dealt with complexity in technological, economic and ecological areas, and proposals were made for overcoming the problems: development of appropriate technologies, more (international) cooperation, generation of public consciousness about these problems, creation of a new ethic. In my view the contribution of the German sociologist Niklas Luhmann (1986) was of special importance. Though his deliberations which were in a certain sense epistemological he substantially enlarged the field under consideration. He

† I am indebted to W. Nemser for going over a preliminary draft eliminating language mistakes and improving style.

said complexity is not only a property of some observed facts – e.g. nature, technology or the economy – but it is also a *property of the observation*, meaning of the relationship between the 'facts' and the observer. In order to understand the results of observations, the *observers must be observed* too. Globally: to the observation of 'facts' by 'society' the self-observation of the society itself must be added.

A first result of this self-observation and one possible explanation of complexity is the fact that society today is not a uniform, homogenous block. Luhmann views it as a framework of relatively *autonomous social systems*. Examples of more or less autonomous social systems are the economy, scientific communities or public administration. The notion of a social system seems to be of high importance in sociology today and for dealing with social development. In order to study social systems some borrowing of ideas from biology and cybernetics, developed there for explaining the nature and functioning of living organisms, has proved useful. Principal concepts are *self-organisation* and *self-reference*, derived primarily from the biologists Umberto Maturana and Francisco Varela. In their view it is characteristic of living systems in biological sense that they are able to reproduce their own organisation (they call them *autopoietic systems*). Varela (1986) asks for the development of a systems theory which is more appropriate for living systems than classical systems theory, which follows the paradigm of dead machines. The main difference is that dead machine-systems can be described by their input/output-relation, whereas for living systems their inner organisation, their inner coherence and their intrinsic structure are relevant. In living systems the interior system of rules – Varela calls it 'operational closure' – leads to so-called 'eigen-behaviours', which are developed without influence from the environment. The interactions with the environment do not take the form of inputs into the organism (for example information) which are processed leading to outputs (observable behaviour), but self-organised systems select from the environment according to their interior organisation, they interpret their environment, they give some things meaning and others not. Varela writes:

> ... for an autonomous machine characterized by its closure and its eigen-behaviour, what happens is that these eigen-behaviours will specify out of the noise what of that noise is of relevance. So, what you have is a *laying down of a world*, a laying down of a relevant 'Umwelt'. A world becomes specified or endowed with meaning; out of eigen-behaviours, there arises possibility of generating 'sense'. So what we are talking about here is the contrast between an instructive Turing automation and an autonomous machine capable of creating (or generating) sense.
>
> (Varela 1986, §. 119)

What is the relevance of these ideas from biology for social systems? It could be that the identity of even such systems has something to do with self-reference, operational closure, and the development of eigen-behaviours. This has always been assumed for most highly organised living beings, for humans. But Varela's deliberations rest on empirical findings concerning cells, nervous systems or immune systems. If even these systems are able to create, through interactions of their parts, such a thing as a collective spirit, shouldn't social systems be capable of doing this also? There are indications that this is the case and this view has been adopted by some sociologists – as already mentioned – as well as by some management scientists. Without going into more detail now – I refer, for example, to Luhmann (1986) and Ulrich & Probst (1984) – I propose for further

deliberation the assumption that the concept of a social system with the attributes self-organisation and self-reference has relevance for human development.

In summing up our considerations so far I would like to give three reasons why I think that self-reflection of social systems as well as of systems of such systems is of special importance in the situation of society today:

(1) to obtain insight into the physical, biological, human environment; for example in order to understand scientific propositions, one has to observe the scientists;
(2) to understand the identity and functioning of social systems, their organisation and eigen-behaviours, to obtain knowledge of their hidden aims;
(3) to arrange, create, change such systems more consciously, and let them die if we wish.

INSTRUMENTALISM AND REFLECTION

Now the question arises: Can mathematics contribute to that process of collective self-reflection? At first glance there are opposed tendencies in mathematics. The most important such tendency is that mathematics is primarily viewed as an *instrument*, a tool, with which humans – as knowing or acting beings – confront themselves with a matter (e.g. nature). With help of this instrument they establish a *distance* between themselves and the matter, they express their differentness. As regards knowledge of nature man has put mathematics between himself and nature – though he belongs to nature too. This kind of process counteracts self-reflection.

Additional support for the view that mathematics is falsely oriented with respect to self-reflection of social systems arises from Varela's confrontation of machine-like and autonomous systems. As an instrument for the comprehension of input/output-relations, as an aid in causing effects on systems from outside, mathematics has so far dealt more with machine-like systems. It is not by accident that mathematical systems theory is sometimes equated with control theory. However, exactly the *statements about the nature of mathematics* I made a moment ago *are themselves reflections*; reflections not only about a certain subject matter but also self-reflections about ourselves, about our relationship with mathematics and those matters to be handled through mathematics. More generally, each instrument which is used by man can be taken as an incentive, a starting point for self-reflection, for it mirrors properties, interests and aims of people – individually and collectively. The more powerful an instrument is, the more potentials for this mirroring it offers.

A second opportunity for using mathematics for social self-reflection comes from the fact that this reflection means a process of communication between people and groups of people. *This communication can be supported by mathematics.* In this case of course the instrumental character of mathematics is of relevance, but with the aim of self-reflection, which influences the kind of tools as well as the handling of them.

In the following I want to propose *two orientations for mathematics* (as regards teaching, research, application), which correspond to the two above-mentioned potentials, and to describe these orientations in detail. They have been included within mathematics all along, but by crude instrumentalism they have been concealed. I call these orientations:

(1) the analysis of basic assumptions of mathematical concepts and theories – mathematics as a mirror of mankind;
(2) problem description instead of problem solving – mathematics as a means of presentation and communication.

ANALYSIS OF BASIC ASSUMPTIONS

Let me start with an example. The following formula is well-known:

$$K_t = K_0 q^t$$

It gives the amount of a capital K_t with compound interest after t years, starting with K_0. Not so well-known is the fact that this formula is the solution of a system of functional equations. Taking K_t as a function of K_0 and t, which means

$$K_t = f(K_0, t)$$

this system consists of the following equations:

$$f(K_0 + K_0', t) = f(K_0, t) + f(K_0', t)$$

$$f(K_0, t + t') = f(f(K_0, t), t')$$

(see Eichhorn, 1978, p. 10).

Both equations can be interpreted as properties of *invariance with respect to splitting*. The first equation corresponds to a splitting of capital, the second to a splitting of time. More exactly: the first equation says, if you split the original capital into two parts, say K_0 and K_0', and given them to two – perhaps different – banks, you get the same result as if you had invested the total amount at one. The second equation says, if you invest a capital first for the time period t, then take the result and invest it immediately for the time period t', you get the same as if you had invested the original capital for the total period $t + t'$. In a certain sense these are the basic assumptions of compound interest calculation. We know that they are not precisely fulfilled in reality. You get a higher rate of interest for a larger capital as well as if you invest for a longer time period.

An interpretation of this example in a social context could be the following: the main purpose of large amounts of capital for long time periods in our economy is *to cause collaboration* in order to reach more productivity than the sum of the productions of individuals would yield. The classical formula for compound interest neglects this purpose. It abstracts on the one hand from the effort necessary to bring people – or capital which is the same in the economy – together, and on the other hand from the fact that if you succeed in bringing people together you can get more than the sum of that individually reachable.

It is well known that mathematics itself is very much engaged in reflection on its concepts, theories and procedures – as in the above example by proving the theorem that the formula for compound interest is the solution of a certain system of functional equations. One can say the theories of applied and even of pure mathematics are results of such reflections. Areas of study include the questions under which assumptions which procedures can be applied, how the results can be interpreted, and so on (think, for example, of mathematical statistics). However there are *differences with respect to the relevance for social self-reflection*.

Take the concept of *number*. One can view theories of the natural or of the real numbers as reflections about the concept of number. The focus is the logical dependence of statements, consistency, relative existence, or proof of non-trivial theorems (e.g. concerning the distribution of primes). Another perspective of reflection on the concept of number, which I view as more fundamental with respect to social self-reflection, is pursued by the *theory of measurement*. This theory is concerned with the question: Under which assumptions it is meaningful to associate aspects of 'reality' with numbers, and how this can be interpreted? A basic assumption of each measuring procedure is the construction of a mapping

$$f : A \to R$$

from a set A of 'real' things into the real numbers, whereby certain relations in A are mapped onto 'sensible' relations in the image set, such as onto the natural ordering or onto addition. A simple theorem of measurement theory about the possibility of such a construction is the following.

Theorem
Let A *be an at most countable set and* S *a binary relation in* A. *Then there exists a mapping* $f : A \to R$ *satisfying*

$$\forall a, b \in A : aSb \Leftrightarrow f(a) < f(b)$$

if and only if S *is a strict weak order* (see Roberts, 1979, pp. 101, 102, 109). (*A strict weak order as an asymmetric relation, the negation of which is transitive.*)

According to the degree of uniqueness of the 'measurement mapping' f different so called *scale-types* are defined – such as ratio scales, ordinal scales, and so on. The theory yields that certain operations with quantities are only meaningful under certain assumptions. To this belongs the well-known fact that computation of the arithmetic mean is not meaningful for ordinal scales.

Though measurement theory remains within mathematical modes of inference and of representation, it also opens up fundamental perspectives. It studies the principles of what we are doing when measuring, but furthermore it poses questions about extensions and alternatives, for instance about non-numerical or multi-dimensional measurement (cf. Roberts, 1979). Here I want to make a very concrete didactic suggestion: measurement theory should belong to the basic education of all mathematicians. Thereby the dominant tendency of theoretical refinement and of the search for interior dependencies with respect to one or more concepts would be supplemented by reflections about their relationship with things outside the world of mathematics, on a level which is more fundamental than treating only some isolated examples of model-building.

The final example for the orientation 'mathematics as a mirror of mankind' that I mention here is the most fundamental: considerations about *connections between the organisation of knowledge and the organisation of society*. The philosopher and social scientist Gerhard Schwarz (1985) interpreted the four axioms of *classical logic*, namely identity, the exclusion of contradictions, the exclusion of the middle and the principle of sufficient reason, as axioms of *hierarchy* as a mode of social organisation. I have extended this interpretation by establishing the 'equation':

logic:hierarchy = mathematics:bureaucracy

(see Fischer, 1987). The analogy is based upon various common aspects of mathematics and bureaucracy, for example autonomous rule system, materialisation of the rules, procedure-orientation. To avoid misunderstandings I point out that I don't have such a negative opinion about bureaucracy as is usually held. For many matters I find it an efficient way of organising social life, in any case it was a historically necessary step in developing modes of organisation. Studying and creating bureaucracies can also be very fascinating -- as fascinating as is the studying and creating of mathematics.

One can write the above equation also in the following way:

logic:mathematics = hierarchy:bureaucracy

Now this means that logic plays a similar role for mathematics as does hierarchy for bureaucracy: both are in some sense the 'organisational skeletons'.

By the way, I recently read about a research result concerning the *problem solving capacity of small groups* which indicates a connection between the kind of information processed and the organisation of a group. If you have noiseless communication about clear matters then hierarchically organised groups get better results (in coordinating information). However if 'noise' is introduced by giving contents which cannot be spoken about so clearly then 'heterarchical' groups are better. They are able to develop a language, whereas hierarchical groups tend to fall apart (see Foerster 1984). Does this tell us anything about mathematics?

PROBLEM DESCRIPTION

Through the following example, taken from a lecture by Fred Roberts (1986), I shall demonstrate the difference between conventional mathematical problemsolving and what I intend. Roberts posed the following problems:

> Given a rectangular street system of a city, each street being two-way. Because of heavy traffic there is frequent congestion and considerable air-pollution. Therefore the government decides to make all streets one-way. How shall it be done?
>
> (Roberts, 1986)

One postulate is of course the possibility of reaching every place from every other place. Mathematically formulated, the arising *digraph* must be strongly connected. However which other postulates are to be posed? What criteria exist for the efficiency of a solution? Roberts made the following proposals:

> For two vertices u, v of the street-system-graph (i.e. crossings) let $d(u, v)$ and $\vec{d}(u, v)$ respectively be the lengths of the shortest connection in the old (undirected) and the new (directed) graph respectively. Consider the following quantities (n = number of vertices):
>
> (1) $\max_{u,v} \vec{d}(u, v)$
>
> (2) $\frac{1}{n^2 - n} \sum_{u \neq v} \vec{d}(u, v)$
>
> (3) $\max_{u,v} [\vec{d}(u, v) - d(u, v)]$

(4) $$\frac{1}{n^2 - n} \sum_{u \neq v} [\bar{d}(u, v) - d(u, v)]$$

(5) $$\frac{1}{n} \sum_u \max_v \bar{d}(u, v)$$

(6) $$\frac{1}{n} \sum_u \max_v \bar{d}(u, v) - \max_v d(u, v)]$$

With respect to each of these quantities one can argue that the smaller this quantity the better the solution. An optimal solution is obtained if this quantity is minimal.

(Roberts, 1986)

Roberts offered solutions for parts of the problem, especially for two of the above criteria for optimality, obtained by sophisticated combinatorial reasoning. This means he presented certain patterns for one-way-street systems.

Now I pose the question: *What does such a solution mean for practical purposes*? Not too much, I think. First, it is unlikely that all streets would be two-way and would have to be made one-way at once, and certainly also not all streets would be equally important. There are districts where people live and others where they work, the density of population need not be homogenous, and so on. This means that none of the given *global criteria* for optimality seems to be totally appropriate. Moreover also other global criteria could hardly do the job. Further it is possible that the people of the city simply *want*, without clear reasons, a certain street to remain two-way. Finally one has to take into consideration the public action of citizens if a 'solution' were to be implemented in reality. (Roberts made similar comments in his lecture.)

Does this mean that mathematics is completely useless for this kind of situation? Not at all! I think that mathematical concepts and modes of representation are in this case appropriate for a *description of the problem*. Undirected and directed graphs offer a visualisation of the situation and can be used for experiments (e.g. simulations). The formulation of the quantities (1), (2) and (3) indicate different points of view in improving decisions. Further concepts can be developed, such as considering the status of streets as main or side-streets according to the present traffic, the system of public transport, the division into districts and so on. For this purpose concepts of graph theory can be used – for example that of flow in a weighted graph – or new ones can be invented. Of course the complexity of the problem would thereby increase and a mathematical algorithmic solution would drift out of reach. (Even for criterion (1) it is a *n-p*-complex problem). However, it would *not* be the task of this process to find a solution. Rather the task would be to give a good description of the problem in order to enable the people of the city and their authorities to discuss the situation and make sensible decisions.

What about the connection with self-reflection? Clearly it is not such a fundamental one as is considering relationships between mathematics and hierarchy, not even as deliberations about basic assumptions of the formula of compound interest. However it is the reflection on the needs and wishes of people in a certain social system, with respect to a given problem situation. Of course reflections of this kind are also included in conventional mathematical modelling. One wants to experience the interests and desires of the people concerned, in order to improve and refine the model. However the goal is

always to obtain a (the) final solution. Therefore those who construct the model emotionally are disturbed by the appearance of new aspects or even of contradictions. They want to be told once and for all what people want so they can concentrate on solving the problems. In contrast to this traditional orientation the mathematician oriented according to my suggestion understands himself as an *explainer of the problem*, who helps people to articulate their imaginations, who points to alternatives; sometimes even as somebody who slows down the process of solving the problem. Whereas in the traditional way of proceeding the mathematician tries to reduce complexity and to exclude alternatives, in this case he acts according to the 'ethical imperative' formulated by Heinz von Foerster (1984, p. 3):

> Act always so as to increase the number of choices.

From various areas of application of mathematics voices can be perceived arguing in a similar direction as I am doing now – though mostly not *as radically*. One area is that of *applied systems analysis* where so-called 'interactive decision-support-systems' and 'strategic models' are intended to serve as means of communication (see, e.g., Dreyfus & Dreyfus, 1986, p. 192, or Meadows, 1986). Another example is that of *measuring economic inequality*. Different measures yield different results, but they can be used as '*prima facia* arguments, permiting situation-specific considerations' as Sen (1973, p. 75) argues. All this can be embedded in some considerations of the psychologist Lee Cronbach about the tasks of research in social science today. They are particularly remarkable in our context since Cronbach was (is?) one of the main advocates for the use of mathematical methods in social sciences. He writes:

> Concepts contribute to pluralistic decision making by helping participants examine their situations and values . . . The social scientist helps not by playing expert but by playing educator, eternally pressing the question, 'Have you taken X and Z into account?' Social science is cumulative not in possessing ever-more-defined answers about fixed questions but in possessing an ever-richer repertoire of questions. The educative influence of a piece of research may extend far into the future. Concepts have enduring value, and so does a sense of what-connects-to-what.
>
> (Cronbach, 1982, § 72–73)

DISCIPLINING AND OBJECTIVATION

The two orientations 'mathematics as a mirror of mankind' and 'mathematics as a means of communication' are concerned with analysis and synthesis of social reality. The first orientation gives more emphasis to *analysis*. By reflecting about given mathematics one tries to *analyse social relations* which are mirrored in this mathematics. The second orientation is more directed towards synthesis. Through communication *social relations are constructed*, social systems with their identity, operational closure and eigen-behaviour are established.

Social systems have always been constructed with the help of mathematics; pedagogues have called the effect of this process on the individual 'disciplining' (*Disziplinierung* in German, meaning 'shaping people like soldiers') and have criticised it. I think that disciplng is inevitable when constructing social systems. I think even that

more disciplining by mathematics will be needed in the future, but that this should be done more consciously and that the means should be more appropriate to the aim of constructing systems the elements of which are humans.

The function of mathematics in the process of collective self-reflection is *objectivation* in the sense of reification. Social and other relations are made objects, and I also mean this literally in the sense of materialisation. Mathematics offers visual pictures, which usually have to be interpreted in a symbolic way, that means for their understanding one has to know certain social conventions. Pictures of this kind are indispensable for communication, if it is to be expanded beyond the scope of small groups. It is not the case that we should use mathematics for communicative self-reflection simply because it is here – though this would be a sufficient reason – but without mathematics, or better without a world of symbols reflecting certain relations, we would be in trouble. The French pre-historian and anthropologist André Leroi-Gourhan even writes: a 'society with declining capacity of creating symbols would lose its capacity for action' (cf. Leroi-Gourhan, 1980, p. 267; see also Fischer, 1984). In this spirit, and considering the potential of computers, I believe that there will be a shift of importance in mathematics from handling given symbol systems towards the *creation of new symbols*, in connection with study of the suitability and social relevance of the symbols. Maybe this is simultaneously a step towards a de-mystification of the symbols. There is always the danger that a necessary objectivation of the abstract leads by mystification to a *Verabsolutierung* (making absolute) of present states. By the way, a very powerful, and a wide sense mathematical, objectivation of abstract issues is *money* – where we face the cited danger every day.

It would be interesting to analyse the ***relationship between disciplining and objectivation*** through mathematics from a socio-philosophical point of view. Objectivation of social relations on the one hand means to establish a distance between ourselves and exactly those relations; thereby it is a means to get free of them. We become able to handle and to manipulate these relations – all this can increase freedom of the individual and of society. On the other hand, objectivation gives the social relations an absolute, maybe even eternal, character – thus disciplining gets more emphasis, it becomes more severe, rigid and oppressive. The construction of social systems with mathematics is pursued in such a way that there is only one possibility, nobody can escape. The 'insight into the necessity' or the lack of it – both can make things repressive for the individual.

I think the way to keep the advantages and to avoid the disadvantages of objectivation is to *make the process of objectivation*, the construction of symbols, *more explicit* and to make it available to people. Thus the objects would lose their absolute, eternal character, it would be seen that there are alternatives. Simultaneously the *construction of social systems and the inevitable disciplining would become more playful*, as is the case for the construction of mathematical systems according to a formalist philosophy of mathematics. I think that one of the problems of social change is that we are too serious about social matters. This seriousness does not allow the flexibility we would need in order to cope with the problems of today. Mathematics is also guilty of this seriousness. On the other hand mathematics would also offer play and freedom. To state it as a prophecy: socially we have not yet attained that degree of freedom and flexibility which we have reached in pure mathematics. As an optimist I say: we are on the way.

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3

Aims and Scope of Applications and Modelling in Mathematics Curriculua

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Among the many issues which deserve attention in discussing the aims and scope of applications and modelling in mathematics education the following are fundamental.

(1) The question of **why**?.
Should, for a given educational level, applications and modelling be part of the mathematics curriculum? If yes, why?

Provided the answer to this question is 'yes', two more questions pose themselves.

(2) The question of **what**?.
What content, products, and processes related to applications and modelling should be made object of study, teaching and activity?
(3) The question of **how**?.
What means would be appropriate for applications and modelling activities, and which are actually at the disposal of teachers and students?

It is to be expected that the answers to each of these questions depend on which segment of the educational system is considered. Let us agree, in this context, to distinguish between three types of mathematics education referring to three different educational levels.

(1) The *general mathematics education* of the population – predominantly supplied by the *school* system – which aims at preparing students for their private and social lives as individuals and citizens.
(2) The mathematical education of people studying subjects or preparing for professions which are not, in a specific sense, mathematical in themselves, but to

which mathematics – by being applied in them – has important services to offer. We shall call people in such professions *users of mathematics in extra-mathematical professions*.

(3) The mathematical education of people who are to enter *mathematics professions* proper, as research mathematicians, as general applied mathematicians, or as mathematics teachers for post-elementary levels (i.e. for secondary school and above).

If we combine the three questions with the three educational levels we obtain the following **question matrix** which forms the focus of attention of the present chapter.

Question / Educational level	Why?	What?	How?
General mathematical education provided by *the school*			
Education of users of mathematics in *extra-mathematical professions*			
Education of *mathematics professionals*			

In what follows I shall attempt to analyse parts of this matrix. Along with doing so I shall give my personal answers to a few of the cell questions. Due to space restrictions, only the Why and What columns will be considered.

THE QUESTION OF WHY?

The most **important arguments** for including applications and modelling in a given mathematics curriculum seem to be the following five.

Applications and modelling should be part of the mathematics curriculum in order to:

(1) foster among students general *creative* and *problem solving* attitudes, activities and competences;

(2) generate, develop and qualify a *critical potential* in students towards the use (and misuse) of mathematics in extra-mathematical contexts;

(3) prepare students to being able to *practice applications and modelling* – in other teaching subjects, as private individuals or as citizens, at present or in the future, or in their future professions;

(4) establish a representative and balanced *picture of mathematics*, its character and role in the world – such a picture must encompass all essential aspects of mathematics, and the application of mathematics and mathematical modelling in other areas *do* form one such aspect;

(5) assist students' *acquisition and understanding of mathematical concepts*, notions, methods, results and topics, either to give a fuller body to them, or to provide motivation for the study of certain mathematical disciplines.

The character and status of these five arguments are not the same. Arguments (1) and (5) relate primarily to educational **tactics**. Argument (1) focusses on formative aspects of the *general education* and personal development of students, not on matters specific to mathematics. Mathematics and application and modelling activities are to serve as a *vehicle* to a general end, rather than to be of independent interest. In aiming at facilitating or improving mathematics teaching, argument (5) is concerned with teaching tactics. *Applications and modelling* form a *vehicle* to this end. The argument would not make sense if *mathematics* were dispensed with.

In pursuing the purpose of preparing students to aspects of life outside mathematics education, arguments (2) and (3) deal with general educational **strategy**. In both arguments students are wanted to come to grips with the actual use of mathematics in the world; in argument (2) in an *analytical* way, in argument (3) in a *constructive* way. To both arguments, mathematics and applications and modelling are essential components, not replaceable vehicles.

Argument (4) concerns students' perception of mathematics as an entity, thus addressing mainly epistemological issues. In contradiction to (2) and (3) which look at the world outside mathematics and views mathematics as a factor in this world, argument (4) looks at *mathematics*.

All these arguments can be encountered in discussions on each of the three educational levels identified in the beginning of this paper. Still, for a given level, some arguments seem to carry more weight in the didactical *debate* than do others. The diagram below summarises this.

	Arguments for Why?		
General mathematical education provided by *the school*	(1)	(3)	(5)
Education of users of mathematics in *extra-mathematical professions*		(3)	(5)
Education of *mathematics professionals*	(3)	(4)	(5)

Apparently, arguments (2) and (4) have not gained much attention. In my view this is not satisfactory. I shall try to explain why, and to establish an alternative priority. A basis for doing so is needed. The decision to include applications and modelling studies and activities in the mathematics curriculum of a given educational level should be derived from a consideration of the overall purposes of *mathematics education* for that particular level.

We will consider **School** mathematic education first. Several reasons for providing the general population with a mathematics education which goes beyond elementary arithmetic with known figures have been put forward through the ages. Frequently, the

utilitarian contributions of mathematics as an instrument for tackling needs relevant to society have served to justify the presence of post-elementary mathematics in the school curriculum. Sometimes mathematics education has been seen as a vehicle for forming and developing intellectual and other mental capacities in people, or as a means for encouraging creative activity among students. Sometimes mathematics has been viewed as a source for aesthetic experiences, or as a testimony of the cultural achievements of mankind. Most often a combination of these reasons has constituted the justification of school mathematics education.

To me the ultimate reason for giving a substantial mathematics education to the general population is that mathematics is being *used* extensively and ever increasingly in society, for better and for worse, in such a way that people's professions and lives as individuals and citizens are strongly influenced by it. The main purpose of mathematics education is to help them become competent, independent individuals in all aspects of their lives, and not victims in their association with mathematics in society. This reasoning recognises the dual interest of a raising a well-educated labour force and in promoting a democratic development of society.

Now, if the mathematics education given in school is to comply with the purpose put previously, it must provide students with prerequisites for understanding, assessing and handling aspects of the use of mathematics in other areas. The use of mathematics is brought about through the construction and application of mathematical models, even if the modelling process is not always explicit, let alone transparent. If we agree that the prerequisites mentioned are not generated automatically by a mathematics teaching focussing solely on pure mathematical concepts, results and topics, we are led to conclude that applications and modelling should be part of the school mathematics curriculum. It appears that we are invoking arguments (2) and (3). This does not imply that arguments (1), (4) and (5) do not have parts to play here, but they are derived arguments of the second order.

The basic reason for providing mathematics education to future users of mathematics in a given **Extra-mathematical profession** is, of course, that mathematics is applied – or applicable – in the preparation for or in the practice of that profession. This is why mathematics is often labelled a 'service subject' to such professions and to studies leading up to them.

At first sight it seems easy to establish that applications and modelling should be included in the mathematics curricula preparing for these professions, simply because of *argument (3)*. I would, however, like to add *argument* (2) to the motivation. The background for this is the following.

When considering the range of areas to which mathematics is a service subject, it is important to keep in mind that the part which mathematics plays in different areas varies very much with the area. We are faced not only with a variation in degree and extent over the disciplines, but also with a *variation in character*.

In some subjects (of which physics is the chief example), mathematics is integrated in the very formation of the basic scientific concepts and theories, many of which cannot even be formulated without mathematics. In addition to this comes the highly complex part which mathematics plays in such subjects in finding solutions to problems, in establishing results, and so on. It is a general feature that in these disciplines mathematics is demanded to provide exact answers or, if this is not possible, exact estimates of errors.

To other disciplines, mathematics is, although highly relevant and useful, not *essential* in the same way as with the group of disciplines mentioned above. In biology and economics, theories and results often exist which make sense and can be formulated without the use of mathematics, but which may also benefit strongly from a mathematical formulation. Another major difference between physics and, say, macro-economics is that physical theories are established through a very elaborate interplay between experimentation/observation and exact mathematical model/theory-building, whereas in macro-economics experiments are not possible in general. So, one would only very rarely require quantitative statements in macro-economics established by means of mathematics to be exact. Rather one would go for and expect qualitative statements. The same holds in biology, even if in biology experimentation is often a possibility. Exact quantitative statements would in general be too much to hope for, mainly because living beings display too complex behaviours.

If the part played by mathematics in academic areas displays a large variation in character, the variation becomes even larger when the full range of extra-mathematical professions are considered. This indicates that the application of mathematics in extra-mathematical contexts is not an unproblematic and straightforward affair. On the contrary, applications and modelling involves practical, scientific, philosophical and even political complications which deserve careful and critical attention. Therefore, the inclusion of applications and modelling in the mathematics curriculum for users in extra-mathematical professions of all kinds should rely also on argument (2). As to arguments (1), (4) and (5) their place in this context is almost identical with their place in the school context, for which reason we shall not comment on them any further.

If, finally, we turn to the mathematics education of **Mathematics professionals**, we encounter a somewhat different situation. To these professionals, the role of mathematics cannot be reduced to its application in other areas. They are mathematicians, whether in pure research, in general applications, or in teaching. By definition, mathematics forms a crucial and indispensable component in their work. In my opinion, however, it is important that they obtain a representative and balanced picture of mathematics in all its aspects, including applications and modelling. Not only do they become more competent research mathematicians, general appliers or teachers of mathematics, if their outlook on mathematics is broader than one having only mathematics as a theoretical edifice in its field of vision, but also their social functions as expert citizens gain in quality if they are educated to become broad-minded professionals with sound judgement, rather than just specialists.

So, I shall invoke first of all argument (4) and secondly argument (2) to motivate the inclusion of applications and modelling studies in the curriculum for future mathematics professionals. Again, some of the remaining arguments have their parts to play. Argument (3) is derived from arguments (4) and (2): if students are to obtain a genuine and substantial impression of the application of mathematics, they must be prepared as if they were to practice it – at least on a small scale – themselves. Finally, argument (5) is relevant in relation to the education of mathematics professionals to the extent that applications and modelling do in fact assist the motivation for or the acquisition and understanding of mathematical concepts, methods, results and topics belonging to that educational level.

Let me summarise my answers to the Why? question by listing my *main arguments* in the Why? column

	Why?
General mathematical education (in the *school*)	(2) (3)
Mathematical education of users of mathematics in *extra-mathematical professions* (2) (3)	(2) (3)
Education of *mathematics professionals*	(2) (4)

Although already implicitly given by the preceding discussion, it should be made explicit that these answers imply, for all educational level, the answer Yes! to the question 'Should applications and modelling be part of the mathematics curriculum for that level?'

If taken seriously, arguments put forward for incorporating applications and modelling studies and activities in a given mathematics curriculum influence *the way* in which applications and modelling are treated in the curriculum and in the mathematics teaching. Applications and modelling work introduced only to motivate and assist students' acquisition of mathematical concepts and so on is likely to differ from application and modelling activities designed to enable students to practice application and modelling in areas outside mathematics. Therefore, the answers to the questions What? and How? must depend intimately, in principle and in practice, on the arguments given to the question of Why?. This takes us to the next section.

THE QUESTION OF WHAT?

What do we mean by an 'application' of mathematics, by 'applying' mathematics and by 'applied mathematics' and 'applicable mathematics'? What do we mean by a mathematical 'model', by mathematical 'modelling', mathematical 'model-building' and by 'mathematisation'? We shall have to content ourselves with giving a few *working definitions*.

Let us imagine an arbitrary area of extra-mathematical reality (a segment of real life, whatever that is, or of another discipline). If the area is submitted to *any* kind of treatment which involves either mathematical notions or concepts, methods, results, topics or theories, we shall speak of the process of **applying** mathematics to that area. For the result of the process we shall use the term an **application** of mathematics.

It appears that the term an 'application of mathematics' represents the broadest possible activation of mathematics to areas outside mathematics itself. The terms 'applied mathematics' and 'applicable mathematics' do not carry independent kinds of meaning in the present paper. When used here, they simply indicate portions of mathematics (from notions to theories) activated in an applicational situation. Thus, 'applied' or 'applicable' mathematics do not refer to specific mathematical topics which, unlike 'pure' topics, are supposed to be of particular applicational relevance.

When a segment of reality is submitted to any kind of treatment by mathematical means, a **mathematical model** is necessary involved. Very briefly speaking, certain objects, relations between them, and structures belonging to the area under consideration are selected and translated into mathematical objects, relations and structures, which are then said to *represent* the original ones. Now, the concept of model can be defined in two different ways. The first possibility is to simply identify a model with the collection, M,

of mathematical objects, relations, structures, and so on, irrespective of what area is being represented by the model and how. The rationale of this option is that a given collection M might serve to model many different areas. The second possibility is to define a **model** as the triple (A, M, f), where A is the segment of reality under consideration, and f is a mapping which translates certain items of A into items of M. This definition emphasises that a mathematical model is a model of something. We shall adopt it in the present paper, which will not prevent us from using more casual phrases such as 'within the model', 'properties of the model' and the like, to indicate matters related to M alone when it is convenient and there is no danger of ambiguity.

By **modelling**, or **model-building**, we shall understand the full process of constructing a mathematical model of a given area. At a minimum, it comprises (*a*) identifying the features of reality which are to be modelled; (*b*) selecting the objects, relations, and so on relevant to this end; (*c*) idealising them into shapes suitable for a mathematical representation; (*d*) choosing a mathematical universe to hold the model (M); (*e*) performing a translation from reality to mathematics; (*f*) establishing mathematical relations between the translated objects, accompanied by assumptions and properties; (*g*) using mathematical methods to obtain mathematical results and conclusions; and (*h*) interpreting these as results and conclusions concerning the original area. In addition the process may include (*i*) assessing the model of confronting it with reality (e.g. observed or predicted data), by comparing it with other models, by relating it to established theory; and finally: (*j*) building, if necessary, a new or modified model, thus running through the stages (*a*)–(*j*) once again.

As regards the term **mathematisation** it is often used as a synonym for 'modelling' or 'model-building'. We shall attribute a more precise meaning to it by defining it to be the translation part of the modelling process, i.e. the stages (*a*)–(*e*) above.

So far in this section we have been exercising conceptual and terminological gymnastics. We shall now address the question of What?.

There are different modes of making applications and modelling objects of study and activity in mathematics instruction.

(1) Students may acquire *knowledge* of
   (a) *existing models* and applications of mathematics – this may include knowledge of different *categories* of models, classified according to mathematical characteristics as well as to characteristics regarding the different applicational areas to which the models refer;
   (b) characteristics of the *modelling* process, either in general or with respect to specific categories of models.

(2) Students may perform *modelling* themselves, either by
   (a) *applying models known* to the students to situations which are new to them;
   or by
   (b) *building new* (to the students) *models*, or *modifying known* ones.

(3) Students may critically *analyse and assess* models, existing ones or models they have constructed themselves with respect to
   (a) their *mathematical* properties (work *inside* the model);
   or to
   (b) the properties, qualities and bases of justification of the models as *representations* of given segments of reality.

If, for our present purpose, we define these properties to form the aspects of applications and modelling, we may phrase the main question of What? as follows.

Which of these aspects should be included in the mathematics curriculum of a given educational level? To answer this question we will have to make use of the arguments which were put forward to answering the question Why? in the preceding section. In that section one conclusion, common to all the educational levels considered, was that applications and modelling should be part of the mathematics curriculum in order to stimulate a critical potential in students towards the use of mathematics in extra-mathematical contexts. It seems pretty obvious that to this end students should acquire experiences in analysing and assessing models and modelling processes which are suitable for their educational level. In other words, aspect (3) should enter into the mathematics curriculum of any level.

Evidently, the capability of analysing and assessing models and modelling must rest on *some* knowledge of models and modelling. Hence if, for each level, mathematics education is to deal with aspect (3), it has to comprise also aspect (1).

Moreover, it was concluded in the Why? section that a further argument for including applications and modelling in the school mathematics curriculum, and in the education of users of mathematics in extra-mathematical professions, was that students should be prepared to practice applications and modelling. Since practising applications and modelling implies performing modelling and since, as previously stated, we hold the opinion that the ability to perform modelling needs educational training, we are led to infer that aspect (2) should also be addressed in the mathematics programmes of those educational levels.

As to the education of mathematics professionals, we shall recall a previous consideration – namely that if students are to obtain a genuine and substantial impression of the application of mathematics outside mathematics itself, they must be prepared as if they were to practice it – to some extent – themselves. So, aspect (2) ought to be part of the mathematics curriculum for mathematics professionals too.

We have, thus, reached the conclusion that *all the aspects here defined should be incorporated in the mathematics education given at each of the three levels*. This does not imply, however, that the aspects are equally important, or that they materialise in the same way, at different levels. In fact they do not. At school level, for instance, aspects (1a) and (2) will occupy a predominant position. In the education of extra-mathematical professionals, (3b) and (2) will be focussed, whereas the education of mathematics professionals will tend to stress aspects (1a) and (3a).

For each aspect adopted in a given mathematics programme several further questions arise. If we take, for instance, aspect (1), 'acquire knowledge of models and the modelling process', it is immediate to ask 'what knowledge?'. Similarly with the other aspects. Naturally, it goes far beyond the scope of a single paper to deal, in a comprehensive way, with all relevant issues of this kind. I shall confine myself to addressing a single point.

In the preceding section it was stressed that the part played by mathematics and mathematical models in different extra-mathematical areas varies strongly in extent and character with the area. The epistemological position of mathematical models varies with respect to several issues.

(i) The *purpose* of models. Is the purpose of a given model to understand features of the area modelled, or is it to create a background for making decisions or taking actions related to that area?

(ii) The sources of model *construction*. For a given model, what are the conceptual and other connections between the components of the model and the constituents (objects, relations, phenomena) of the area it is supposed to model? How, further, is the model related to theory concerning the extra-mathematical substance of the area modelled?

(iii) The nature of the *statements* produced. Are the statements generated by a given model deterministic or stochastic, quantitative or qualitative, exact or approximate?

(iv) The *verificational status* of models. To what extent is a given model reliable? What is its explanatory value? On what foundation can we hope to tackle such questions? Does the model reproduce known data or phenomenae? If yes, within what range of cases? Is the model corroborated by, or at least not in contradiction to established theory of the area at issue? Is the model related to other models in analogous situations?

Unfortunately, we have to abstain from exploring in any detail examples which may illuminate the outlined epistemological variation of mathematical models. Let me just mention that the way in which plane triangles and their properties in Euclidean geometry serve as models of certain earthly physical objects, and the way in which the logistic growth model represents biological populations, are of a quite different nature.

Similarly, the models from classical mechanics used to describe satellite orbits differ fundamentally from Poisson processes taken to model the number of customer arrivals to a supermarket in certain time intervals. If the triangle and the satellite models were overruled, entire established theories would break down, not just the models. If, for a specific biological population, the logistic growth model failed to hold, well, that model simply has to be replaced with a different one, an event which would not shake the scientific foundation of anything. The same is true with the consumer arrivals model.

Another kind of epistemological position is held by plane map projection models of (segments of) the surface of the globe. Owing principally mathematical reasons, such models can never reflect all geometrical features of the globe, let alone the planet Earth. However for a given projection we are able to tell in exact terms how an area of the sphere deviates from its projected picture. Still another situation is encountered in financing. Though it is perfectly possible to model correctly and exactly the amortisation of an annuity loan in nominal terms – because the concepts involved have already been given a definite mathematical form – the situation changes if we want to model the amortisation in real terms, where inflation is involved. By using different inflation assumptions it is possible to model different scenarios to approximate the real course of events.

Numerous examples could be given in the same vein. In fact, every model may contribute to illustrate the points made here.

For all three educational levels I claim it to be important that students *acquire knowledge* of the described *variation* in *epistemological status* of mathematical models. Such knowledge (which belongs to the scope of aspect (1a)) should be based on a number of characteristic model cases, different from level to level. Another difference between the educational levels may consist in a varying emphasis on the four epistemological facets. While at school level, and in the education of mathematics professionals, particular attention would probably be paid to the source of model construction and to the statements produced by models, the verificational status of models in relation to their purposes is likely to be in the focus of attention in the education of users of mathematics in extra-mathematical professions.

CONCLUDING REMARKS

In recent years much attention has been devoted to introducing applications and modelling in the theory and practice of mathematics curricula. Papers and books have been written, and conferences have been held. Experimental educational programmes have been devised and some carried out. Teaching materials have been developed. Ideas, materials and opinions have been exchanged. In spite of the fact that all this has taken place at the frontier of mathematics education in theory and in practice, rather than in common day-to-day mathematics teaching, I think it is fair to speak of a successful movement of applications and modelling influencing mathematics education at all levels pretty strongly. It is not unusual with movements in progress that more attention is paid to the promotion of ideas and messages than to analytical reflection. In my opinion, the applications and modelling movement has reached a stage where a foundation for it in relation to mathematics education is needed. It is far from being claimed that such a foundation is provided by the present chapter. It has been possible, here, only to sketch elements of an analysis. One point deserves to be emphasised: the aims and tasks of *applications and modelling* in mathematics curricula should be viewed as forming part of the aims and tasks of *mathematics education*, which in turn form part of the aims and tasks of *education* as preparing its receivers to live and act in *society*.

Looked upon from an appropriately distant point of view, the ***ultimate raison d'être*** of exercising mathematics and mathematics education at ***a large scale*** in society is that mathematics is important in the pursuit of extra-mathematical purposes, brought about through its application to other areas. Therefore, no main level of mathematics education should be devoid of an applicational perspective. So, it was concluded in the preceding sections that basically the same arguments for including applications and modelling studies in the mathematics curriculum are valid for any educational level, from school to university mathematics, and that the same aspects of applications and modelling ought to be dealt with at all levels. What varies between the levels, and between different programmes, is the way in which these aspects are *realised*.

Models and modelling may be made the objects of study, teaching and activity in a multitude of ways. In conclusion one thing should be kept in mind: models are designed to model *something* - not to be confused with something *unique*. For a true perspective on applications and modelling to be generated in mathematics education, at whatever level, not only the mathematical components need proper attention. Extra-mathematical substance has to be taken seriously too. Certainly, artificial applications and modelling situations may possess great educational value, but sometimes mathematics education must confront students with the exciting and mysterious meeting points between mathematics and reality, so difficult but so important to understand.

4

Recent Applications of Mathematics and their Relevance to Teaching

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I should like to begin by quoting a story which has just been published in *The Newtonian Revolution in Science and Its Intellectual Significance*, by I. Bernard Cohen, Emeritus Professor of the History of Science at Harvard University. It was published in 1987 by the Burndy Library in Norwalk, Connecticut. What follows is partly paraphrase and mostly direct quotation.

Prof. Cohen begins his paper with this anecdote attributed to an earlier professor of the History of Science at Harvard, Professor Joseph Lovering. Professor Lovering was

> . . . famous a hundred or more years ago for a lecture he gave all over New England. The subject of the lecture was the end of the world. With great enthusiasm he used to detail how a distant astronomical body would come closer and closer to Earth. During this process our Earth and its atmosphere would get hotter and hotter and, eventually, all life would end as a result of the intense heat.
>
> While his audience was still reeling under the impact of this spectre of death by fire and heat, he would turn to them and say: 'Now let us discuss the probability of this event.' According to the theories of the great Laplace, he would conclude, the probability of such an event is so small that it may be considered impossible.
>
> In a day when there were no television sets, no radios, and no motion pictures, people found this lecture a thrilling experience. There was always a good attendance when Joseph Lovering gave his presentation. He really deserved his fee of (perhaps) forty dollars.
>
> Once a group of Harvard alumni asked Lovering if he would come to Plymouth to give his famous lecture. Unfortunately, they could only pay twenty dollars; Lovering told them his fee was forty and he wouldn't come for less. The alumni group appealed to President Sparks of Harvard. They said they didn't think this was the proper

attitude for a member of the faculty. The president intervened and, in the course of time, the professor went to Plymouth and gave his famous lecture.

..Lovering was never in better form. He became more and more eloquent as he described how the temperature of the earth became hotter and hotter. After he had described the heat-death, with the earth all in flames, he concluded abruptly with a 'thank you', and sat down.

The president of the Harvard almuni group was very uneasy as he looked at the disturbed faces in the audience. He jumped up and said, 'Prof. Lovering, you can't do this to us! Don't you usually talk to us about probability? Can't you give us some assurance that this won't happen to us?' Whereupon Lovering, being a good Yankee, replied, 'Not for twenty dollars.'

Now let me spoil this story, and take all the fun out of it by analysing some of its implications. First of all, it reminds us of the fact that classical applied mathematics, that is mathematics applied to classical physics, is only a part of the story. Probability, and other branches of mathematics applied to human society, is another part of the story. Both are essential aspects of applications of mathematics, and of model-building. You only have to look at the diversity of the program of this conference to see how far we have come. We have mathematics applied to physics and chemistry and biology and communication theory, to geology and epidemiology and oil exploration; we have mathematics applied to strategy and speech and risk and optimal search and sport and interest rates and automobile traffic and the Suez Canal – and war and peace! We have applications of everything from ancient geometry to fractals.

If you want to explore further the tremendous variety of mathematics which is applied, and the variety of fields to which it is applied, you need look no further than the work at the University of Kassel by Professor Blum and his colleagues. Their bibliography, for example, is a model of what can be done in this field.

Let me come back to the story. The audience at Plymouth were graduates of Harvard University who paid to hear a lecture on science. A hundred years ago, they had not learned any probability at Harvard. (Forty years ago, I didn't learn any probability at Harvard either!) So they didn't know how to think about very low probability events of very high risk. There are nowadays, many more low-probability high-risk potential events for us to worry about than there used to be. *All* students should now learn probability – and not just for that reason. They need to begin the study of probability in elementary school, and to continue it for many years afterwards. Pioneering work on the early introduction of probability was done in Scandinavia in the 1960s, in Hungary, and by Arthur Engel here in Germany.

The story also reminds us, rather superficially I admit, of mathematical economics, and thus of mathematics applied to the social sciences more generally. There is not as much at this meeting on this area as one might perhaps have expected – but you can't have everything. Mathematical models in economics, in individual and group behaviour, and in political science, are also terribly interesting.

Why do we teach mathematics anyway? Professor Blum reminded us of some of the reasons when he opened this conference, and I shall summarise them in my own way once again. I think there are four major purposes inherent in the teaching of mathematics:

(1) mathematics for practical everyday life;
(2) mathematics for intelligent citizenship, and for the decisions that must be made in life – but not every day;
(3) the mathematics needed for your profession, and for the preparation for that profession;
(4) mathematics as part of overall human culture.

Mathematics for practical everyday life is traditionally the responsibility of the elementary school. Mathematics for intelligent citizenship – and we shall discuss this considerably more in what follows – should be the responsibility of the secondary school. At least, that is my opinion. Mathematics for your profession is the responsibility of tertiary education, at least for those professions which require advanced education; otherwise it is also a responsibility of the secondary level. To teach mathematics as a part of overall human culture is, as far as I can see, nobody's specific assignment. Professor Steen likes to argue that this should be the responsibility of graduate school – to prepare at least some highly educated people outside of mathematics to be able to understand to some extent what a new development in mathematics might mean. He points out that mathematics is almost a third culture, in the sense of C.P. Snow's two cultures: humanism is to science as science is to mathematics! In other words, new results in mathematics are yet another order of magnitude more difficult to appreciate than new results in science, and scientists sometimes find it as difficult to understand new mathematics as humanists find it to understand new science.

Twenty years ago, there was a conference in Utrecht, organised by Professor Freudenthal, entitled *How to Teach Mathematics so as to be Useful*. What is being said at this meeting in Kassel that could not have been said in 1967 in Utrecht? Why do we need to keep having meetings on mathematics education? Can you imagine such a question being asked about an International Congress of Mathematicians? It would be totally absurd if a meeting is devoted exclusively to mathematics research – every paper would contain something brand new, in addition (we hope) to some historical background and motivation for the problem under consideration. However, at meetings devoted to education, you may well hear a paper that could have been given twenty years earlier. What is it that changes in mathematics education, what makes new work important and necessary? I should like to mention three major influences: technology, applications, and mathematics itself. All three keep changing, and this fact above all others continually forces us to rethink what we teach and how we teach it. Our knowledge about how children learn, about the psychology of teaching and learning, also keeps changing, but that is not the subject of this talk. We do, however, discuss the teaching of modelling.

Technology, its continual change, and the effect of this change on mathematics education, are an essential part of the current scene, and will play an important role in what we have to say. It would take us too far afield, however, to concentrate on this issue. For some observations, see, for example, Pollack (1986).

The continual change in mathematics is also worthy of much more emphasis that we shall give it. Think, however, of some of the big events, the kind that make newspaper headlines, in mathematics in recent years – since the middle 1970s, let us say. The four-colour problem was solved, with the proof using the computer in an essential way. The stable inversion of the Radon Transform was accomplished; it is better known as computer assisted tomography, and earned the Nobel Prize. CAT scans have been

followed by NMR and PET, which involve mathematical problems of a similar spirit. Linear programming made the headlines, first for the proof by Khachian that the problem is indeed polynomial, and then for the ideas of Karmarkar that enable us to solve problems an order of magnitude larger than could be considered previously. Phenomenal new advances in knot theory are being applied to DNA. Public key cryptography was invented, and complexity theory and number theory were used to show that it was possible. Much more recently, even more ingenious number theory was used to break one of the methods of public key cryptography that had become a favorite among practitioners. Finite simple groups were finally classified completely – there are important relations to coding theory there.

Those are some of the big events in mathematics in recent times, and we notice how strong the applied flavor is to almost all of these. Even the four-colour problem has long been known to have strong relations to switching theory. However the whole body of mathematics mirrors some of the key features of the headline-making examples we have mentioned: mathematics has become much more algorithmic, so the question of how you actually carry out a particular process has become more significant. The nature of proof, and of certainty more generally, has expanded – not without controversy, as can be seen from the debate on the four-colour theorem. Computer graphics has come to play an enormous role in both research and teaching. Above all, we have rediscovered the fact that mathematics is an experimental science, and we are admitting that to our students, for the first time in many decades. All of these changes, as we can see, are closely tied to both applications and technology, but they are important changes in mathematics itself.

Let us now return more specifically to applications and to modelling, which are the focus for the rest of my thoughts at this time. I come to this meeting with a mixture of backgrounds. I worked for 35 years in industry, both as a mathematical researcher and as a leader of mathematical research in the telephone industry in the US. What did I see as the most important areas of research in the mathematical sciences in industry – at least the communications industry? Here are some of the outstanding areas: optimisation; exploratory data analysis; discrete mathematics – i.e. graph theory, combinatorics, information theory, and coding; algorithms – for example computational complexity, and the theory and practice of distributed computing; systems analysis and systems thinking; computer communication, especially mixed voice and data and video traffic; artificial intelligence, including the problems of human interaction with machines in an intelligent way.

These are the areas which were most important for mathematical research in the telephone industry in recent years. What were the areas which arose most frequently in the advice we gave those of the approximately one million employees of the Bell System (before divestiture) who came to us for help? We probably got more questions involving data analysis and statistics than any other single field – with discrete mathematics a close second.

Probability, especially involving traffic theory, was often called upon. So were other parts of operations research, communication theory, and algorithms. These are some of the main areas which employees of the Bell System needed most – and which very few of them had ever had a chance to learn! These areas are required for so many occupations, and so many applications, that are important for industry.

It is only natural that there should be a very great overlap between the areas of

mathematics most important for [illegible] in industry, and the areas not in their educational background which were most useful to the totality of employees. The amazing fact is that these same areas come up in yet a third line of reasoning, one which we began earlier in this presentation and to which we now need to return. We said that one of the major purposes of education in mathematics is to provide the mathematics needed for intelligent citizenship and for personal decision-making. We also wish to define this as the major function of mathematics in the secondary school. What is wrong with the population's intelligent citizenship and decision-making now? Why does anything need to be changed? Here are some troubling observations.

People don't seem to be able to handle uncertain situations – those which are inherently stochastic in nature. For example, as we saw at the beginning of this presentation, low-probability but high-risk events are very difficult for many people to consider in a rational way. People have difficulty reasoning from data. In fact they are afraid of data, and will tolerate outrageously misleading graphics in popular and even technical presentations. They cannot plan intelligently – never mind optimise! The notion that when you consider alternatives, you must think about your optimisation criterion is indeed difficult for most people. On a somewhat simpler level, but still dealing with planning, how many people can [illegible] alternative life insurance plans, financial possibilities, or interest rates? And [illegible] the inability [illegible] think about systems. People always look for simple solutions: 'throw more money at the problem', 'twice as much is twice as good', 'shoot'. They [illegible] to behave linearly, there are no unforeseen consequences, no instabilities, feedback loops and no overloads. They don't think discretely or algorithmically at all, they do not know that it is possible to model situations in the real world in a structural, quantitative, analytic, that is mathematical way.

We are discussing the importance of including probability and data analysis, planning and optimisation, modern finance, discrete and algorithmic mathematics, and even systems simulation in the secondary school curriculum. Above all of these looms the need for learning something about modelling, and to have some experience with it. It would be well for all countries to think about these things: What is most important for all students, for all future citizens, to know? This meeting will provide much food for thought for these endeavours.

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5

Applied Mathematics Instruction in the Lower Secondary Level – Between Traditional and New Approaches

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This chapter will deal only with the situation in Germany after 1945, particularly in the FRG.

1. TRADITION

Traditionally there are two general directions connected to the **Gymnasium**, the other connected to the *Volksschule*. Although there have been several attempts (Klein, Lietzmann) to move near to the two directions, although a third kind of school was established inbetween (*Realschule*), they developed independently on purpose.

1.1 Gymnasium

Until the early nineteenth century, applications were an integral part of mathematics education. During the course of that century, however, the applied contents and methods were increasingly reduced, for several reasons. *Neo-humanistic* ideas about education included a kind of *horror utilitatis*, generative educational goals became more and more important and the teaching of mathematics had to be legitimised in an appropriate manner.

Since about 1870 slight, and later rapidly increasing, *opposition* arose against this reduction. This opposition has its roots in the immense progress of applied mathematics and, even more, in the importance of technology, with its fundamental use of mathematics. The opposition is mainly expressed by the newly formed technological intelligentsia (*Vereine Deutscher Ingenieure*), but also by teachers (Inhetveen, 1976).

In 1904 the *Gesellschaft deutscher Naturforscher und Ärzte* established a committee, which in 1905 proposed the *Meraner Lehrplan* (the Meran directions). These recommendations shaped the teaching of mathematics in the Gymnasium until the 1960s.

On revising the Meran directions in 1922, the *Deutscher Ausschuss für den mathematischen und naturwissenschaftlichen Unterricht* (*DAMNU*) pleaded for an even greater emphasis on the significance of mathematics to the real world. In 1923 Zühlke published his book *Politische Mathematik* and stressed therein the civic duty for applied mathematics teaching (which seems to have been self-evident to the reformers at that time); in 1924 there appeared Lietzmann's (Klein's successor as the head of the reformatory movement) *Methodik der angewandten Mathematik*, the third and last part of his comprehensive *Methodik des mathematischen Unterrichts*. Thus we should be able to assume that applications had become definitely and sensibly an integral part of mathematics education.

Things went the other way, however. Already in 1925 the directions for the Prussian Gymnasien (the so-called *Richertsche Richtlinien*) contain weakening arguments: in the foreword there are to be found several warnings not to neglect the true(!) arithmetic when dealing with applications, and in the curriculum itself there are fewer suggestions for applications than in the Damnu directions.

Further on there is much evidence that at that time teachers actually took up only a very small proportion of these suggestions.

Lietzmann himself seems to retract some of his former statements (whose character had been a kind of compromise between formative and utilitarian goals) (Kaiser–Messmer, 1986, pp. 36ff). In the 1953 edition of the first part of his *Methodik* he finally states 'that real world examples are not to be dealt with for their own sake but for the sake of mathematics'.

In general we can say that, *after 1945*, on the one hand the tradition of the Meran reform was continued and on the other hand that component which put more *emphasis on the relation to the real world was neglected*. This new displacement of applied mathematics teaching in the Gymnasium (after the one in the nineteenth century) depends on several reasons including the following.

- Educators are disappointed about the perversion of applied mathematics teaching which had been part of the Nazi educational policy.
- The weekly number of lessons on mathematics in schools has continuously dropped during the twentieth century.
- In lower classes mathematical content is often taught merely as a provision for its use in higher classes.
- The curricula of the various school subjects are harmonised in an insufficient manner.

However the main reasons are as follows.

- Mathematics teachers at the Gymnasium see themselves as guardians of the science of mathematics (Schefer, 1969), but not of its real world applications.
- The educational idea of the Gymnasium is still adjusted to the neo-humanistic idea of man, neglecting social manners (Schefer, 1969) and optimistically trusting in the effect of formative goals and in an automatic transfer to the world outside school.

1.2 Volksschule (and Hauptschule as successor)

Of course, things were much easier for this type of school. It has never been tied to sciences or even to their systems (Flitner, 1954). Pestalozzi already indicated the main

principle which has shaped the *Volksschule* until now (in particular by means of the so-called *Reformpadagogik*): the principle of *Lebensnähe* (close connection to real life). It is the starting point for the justification and development of the so-called working school (*Arbeitsschule*), school of native region (*Heimatschule*), school of true life, of project-work and various other ways of dealing with content belonging to several subjects from inside and outside school (*Gesamt-, Ganzheits-, Epochenunterricht*). These different approaches include partially different aspects and sometimes contested interpretations of what 'close connection to real life' should mean: rooted in the child's life vs. preparation for vocational life; practical proficiency vs. artistic creativity; intuition vs. analysis and so on. However, they all have in common the attempt to bridge the gab between school and the real world. In the teaching of arithmetic these ideas produced a remarkable phenomenon: the *Sachrechnen* (practical applied arithmetic). This stands for solving simple problems of everyday life by using simple mathematical methods. In the *Volksschule* this *Sachrechnen* was definitely the culmination of all arithmetic endeavours, whereas in the Gymnasium the corresponding *bürgerliches Rechnen* (middle-class commercial arithmetic) was treated as a staging post, to be passed through necessarily, and soon to be left behind on the way to more essential insights.

The undisputed basic principle of arithmetic teaching in the *Volksschule* read like this: 'All insight and all training starts with genuine real world situations and leads back to them.' Over several decades a methodology was constructed for the customary ways of calculation: with percentage, interest, ratios, mixture ratios, and so on. The culmination and conclusion of this principle takes place in Oehl's book, *Rechenunterricht in der Hauptschule* 1965, described by him in Fig. 5.1. This tradition has encouraged some important *insights* which we do not want to miss now.

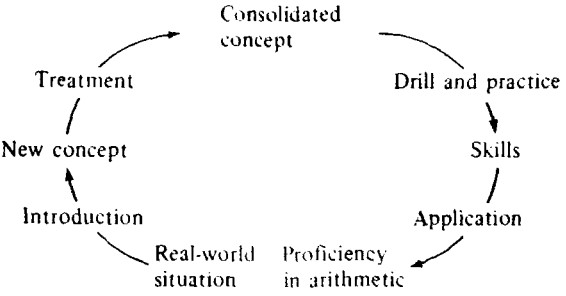

Fig. 5.1

- The differentiation between skills (*Fertigkeit*) and proficiency (Fahigkeit: being able to use one's skills in real world situations effectively).
- The differentiation between the level of the real-world situation and the level of calculation, and the necessity for changing the level twice when solving a real-life problem.
- The right of the real-life situation as an object of analysis on its own.
- The need for genuine real-world problems instead of mere word problems.

There are also two *points of criticism*:

(1) Educators weren't able to withstand the temptation to postulate too quickly (a) the way of life the students have now and in the future, (b) the kinds and places of situations which arise in their lives and can be easily grasped by number, measure and shape, and (c) the methods for teaching the students in advance how to deal with such situations.

These postulates have been questionable even in times where things did not change noticeably during one lifetime, because their actual function was to support the stabilisation of the social strata with their different levels of the quality of life. In our far more complex and fast-living world these postulates would be downright precarious. In the middle of our century, it became clear that these mathematical (and sub-mathematical) methods taught in the *Volksschule* were insufficient in their essence and extension (Winter 1972; Meyer-Drawe, 1981).

(2) Since the days of Pestalozzi and Herbart an important part of the discussion about the education in the *Volksschule* consisted in a search for the 'right' method, the 'sure' way, the 'normal' procedure. This dominance of methodology has impoverished applied mathematics teaching in two ways.

(i) Applications always appear in the same style and with the same relativised, even apportioned, meaning. There is no place for a method of teaching which has phases where relations to the real world are constitutional and essential, and phases where these relations do not play any role.

(ii) Using application as a methodological aid promotes the embarassing qualities

simple = concrete = intuitively clear = close to everyday life.

In order to maintain this fiction, one has to avoid complex applications and restrict oneself to the retranslation of word problems, which have been produced by merely wording arithmetical relations beforehand.

2. NEW APPROACHES

The existing state was then influenced by two new trends, one following about ten years after the other.

2.1 New Math

Today it is nearly forgotten that New Math emerged for solid *economical reasons*. The consequence of the Sputnik shock in 1957 in the western world was the fear of falling behind the eastern world in the political and economical sphere, connected with the hope of preventing this decline by a modernisation of the educational system, mainly in the mathematical–scientific field, and particularly by adjusting the teaching to the recent progress in the corresponding sciences. In Europe from about 1960 the OECD started work by organising a lot of conferences and releasing numerous publications. In our

discipline there was the hope that after the Bourbaki reform for the first time the fundamental ideas of mathematics would become accessible in an elementary way. Reformers in the schools were fascinated by the possibilities which seemed to arise from this development. Their work at the famous conceptual frame 'set-function-structure' seemed to be justified by results from cognitive psychology (Piaget) and produced some success (Dienes), which even surprised the (few) sceptics. That is why soon *Real-* and *Hauptschule* and primary schools in particular were included in this work.

However, it soon became clear that this approach, originally well founded in practice, *altered* in the course of the 1960s: the means (structural clearing up, logical foundation, lingual exactification) became the ends.

The place of genuine applications of new content and method was taken more and more by artificial constructions (e.g. the logic blocks) whose purpose was just to serve as models (in the mathematical sense) for the structures to be acquired. Meschkowski (1965, p. 181) stresses that a direct help of mathematics education for the practice is not necessary: 'Nothing is more practical than a good theory.'

In the curricula, this corresponds to the fact that applications appear only in the forewords. The discrepancy between the original intuitions and concrete contents shows very clearly, for example, in the *Nürnberger Rahmenplan* by the MNU (Association for the Support of Mathematics and Science Teaching) in 1965 and in the *KMK-Richtlinien* (Directions by the Conference of Ministers of Education (each of the 11 states of the FRG has one)) for the modernisation of mathematics education in 1968, which were strongly influenced by the *Nürnberger Rahmenplan*. In the *Richtlinien* one can read again and again about the necessity to grasp the real world with the help of mathematics, to solve concrete problems, and so on – arguments for the justification of that modernisation. However, in the directions themselves one can find almost no hint; the contents for *Real-* and *Hauptschule* originate from the instructions for the Gymnasium by merely omitting parts.

From about 1970 there has been sporadic, and since 1975, a broad criticism against New Math – against several of its goals themselves and against the perversion of others in teacher education, school textbooks and practical teaching. Meanwhile this approach has disappeared to a large extent, and one can state that *New Math* certainly *did not promote applied mathematics teaching.*

2.2 Application-oriented mathematics

In close connection to this criticism against New Math, educators started to consider questions such as how to deal appropriately with the relation between mathematics teaching and the real world, and how to teach the fundamental goal 'ability to apply mathematics', questions which were still open. These considerations emerged worldwide and influenced the discussions in the FRG (as New Math had done previously). The gradual advance of stochastics (statistics) went well with this trend; at first it penetrated the upper secondary level, where it could still be taught as a mathematical theory, but then, from about 1975, also the lower secondary level, where stress had to be laid on reasonable action and behaviour in stochastic situations (Engel, 1976).

From then until the present many concrete examples for applied mathematics teaching have been introduced, most of which have been tested in the classroom (e.g. Becker, 1979; Becker, 1983; Schmidt, 1984; Mued (*Collection of Mathematics Lessons*)). Also

fundamental considerations on the intentions, possibilities and limitation of such a kind of teaching, and its compatibility with other fundamental goals have been developed (e.g. Burscheid, 1980; Weber, 1980; Beck, 1982; Blum, 1985). During the times of New Math pedagogical activities of this kind and quality had been rather rare.

Of course, one can find different positions in this new and continuing discussion. In her recently published thesis Kaiser-Messmer gives a detailed and profound insight into the history and the present state of applied mathematics teaching (Kaiser-Messmer, 1986). She distinguishes between three main lines – emancipatory, science-oriented and integrative – but she also points out that these lines are moving towards each other. In fact, today we see mainly what they have in common compared to New Math and we see the progress compared to historical approaches. I now analyse this in more detail.

(a) At first there is the conviction that solving an extra-mathematical problem by mathematical means is equivalent to constructing a mathematical model which, furthermore, is not inherent in the original problem situation, but has to be added to it.

For example, the price for 5 kg of grapes is not necessarily five times the price for 1 kg of these grapes, but depends on whether the grocer decided to sell his grapes 'proportionality'. The progression from the price for 1 kg to that for 5 kg is part of the model. This was remarked at first by Kirsch in his *Analyse der sogenannten Schlussrechnung* in 1969. Whereas Oehl still asserted that the student has to discover the mathematical relations hidden in the problem situation, we nowadays face this pedagogical Platonism in an extremely critical way. We tend to characterise man as a modelling being (Stachowiak, 1973).

- When we proceed from objects and their connections in the real world to mathematical concepts and their relations we work with images of this real world.
- These images are reductions of the original objects covering only the relevant characteristics.
- This relevance is a feature of the subject who constructs the model.

Of course, this threefold relativisation not only includes but even requires the existence of some trans-subjective standard models. As a rule these are models which come into question in many simple situations, for example the basic foundations (which are taught at the lower secondary level, but not with due regard to this aspect).

(b) Application strictly speaking (i.e. using a model already in existence) and mathematising (developing a useful model) must necessarily be a *process*. A rather rough model for this process, concentrating on widely accepted components, is shown in Fig. 5.2.

Let us consider an example for this control circuit. The situation: an election for a parliament has taken place. We know the distribution of the votes to the parties and, of course, the total number of seats. The problem: how to allot the seats reasonably to the parties? We first use the standard model proportionality and therefore have to distribute the seats according to the ratios of votes. This simple innermathematical 'solution' shows to be useless when retranslated in the situation because the calculated numbers of seats – as a rule – are no integers. We have to improve the model. Thus we come to the competing and not at all equivalent procedures after Hare and D'Hondt. Which is the more reasonable one? To answer this question, one needs not only mathematical but also political criteria.

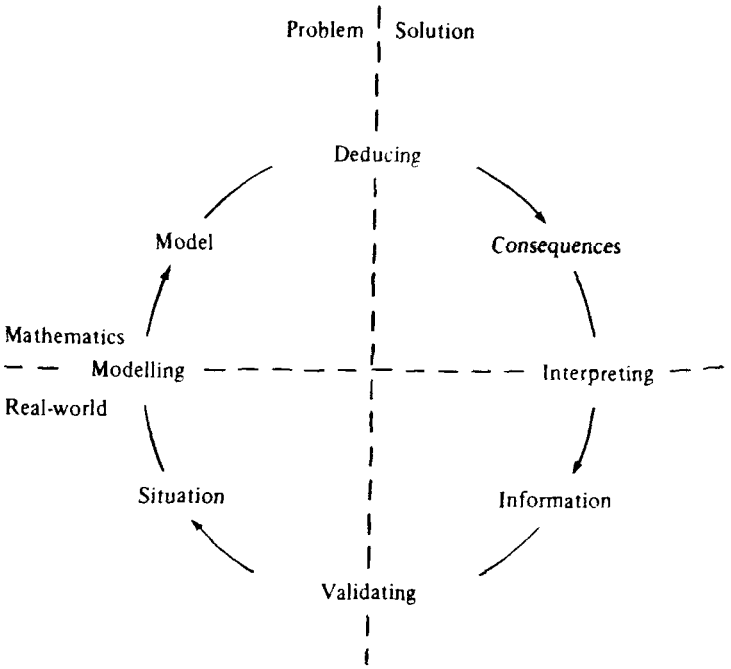

Fig. 5.2

Of course, it depends on the kind and on the complexity of the given situation, how often we have to go round the circuit and how important are its single steps and stations. However we should never neglect to validate the information we have got by modelling (as it is usually done in classroom). The students must learn to use non-mathematical arguments in this phase, too.They should also be confronted with problems to the solution of which mathematics can only contribute – more or less.

(c) We are convinced of the need *to make* those processes *explicit for the students* and that this must be done by passing (and repassing) around the control circuit of Fig. 5.2 knowingly. The arguments for this are (among others):

- to instruct the students in solving those tasks completely;
- to support a comprehensive understanding of the situation, a sensible choice (or construction) and use of a model, and a well-weighed decision;
- to prepare the students for applications outside and after school.

From the present point of view, in order to treat practical problems reasonably, one has to possess a well-founded body of relevant knowledge about the contents of a situation as well as about the necessary mathematical methods, or one has to acquire it spontaneously.

3. THE TEACHING OF MATHEMATICS

3.1 The situation

Was the wave of New Math followed by the wave of applied mathematics teaching? If yes, then this new wave, unlike the first one, did not run through the schools, but was restricted to the pedagogical discussion and the work of some motivated teachers. In the *Hauptschule* calculations with measures and *Sachrechnen* are still prevalent, and in the *Realschule* and the **Gymnasium** one goes on retranslating word problems into mathematical relations. We have to ask anew which are the causes for the small effects in practical applied mathematics teaching despite all our sublime goals and various efforts?

The following reasons (which we add to the reasons sketched in 1.1 and 1.2) seem to be independent of the type of school:

(a) Applied mathematics teaching is very ambitious – for teachers and students and nowadays even more (and not at all motivating by itself from the first).
(b) The possibilities for helping the teachers from outside (directions, books, and so on) are limited. The ways in which genuine real world problems are posed, treated and discussed in the classroom depend strongly on the particular conditions of each class.
(c) Unlike mathematical problems, real life problems are not perfect. What is given, what is wanted, which means are permitted, need not be clear, but can be a part of the problem, even its cause. Thus the arguments and answers are subjective, ambiguous and provisional – compared with innermathematical solutions. Teachers obviously prefer the latter ones (especially when they dominate at university as is the case).
(d) The organisation of the educational system on the whole and of the teaching units in particular, especially the so-called *Aufgabendidaktik* (adjusting mathematics teaching to solving many small problems (Lenne, 1969, pp. 50ff)), produces an unfavourable frame for applied mathematics teaching.

3.2 Possible changes

If one is convinced that applications must be a necessary component of any mathematics education, so that

- students can move confidently in the fundamental fields of everyday life, where simple mathematical methods can help (utilitarian goal)
- they are equipped with the long-term ability to behave reasonably in complex situations as well (formative goal)
- they eventually get an appropriate view about the essence, potential and limits of mathematical methods (methodological goal)

then the question arises how to effect a distinct change in the practice of teaching.

This question is – of course – directed at the mathematics educators, the didacticians. They are not relieved from their responsibility by merely having propagated, substantiated, analysed and exemplified applied mathematics teaching for years. They have to come more into contact with the teachers in the schools; they should plan, carry out and evaluate appropriate short- and long-term projects together with them (an

example for such a cooperation is given by Jager & Schupp, 1983). However the question is directed above all at the mathematics teachers themselves. What can they do now and under the given conditions?

The following ideas are not meant to be recipes, but hints for more competency in mathematics application, which can be used immediately. Spectacular suggestions such as project-work or team-teaching are not included, because I think ideas for everyday teaching are more likely to be realised.

(a) Those parts of the curriculum should be extended where applications are constitutive (see Kaiser *et al.*, 1982) at the expense of other sections which merely provide techniques for situations still (or never) to come. I mention above all stochastics, but also parts of geometry (e.g. shape, genesis, function and construction of everyday objects, measuring of inaccessible lines and angles) and of algebra (e.g. empirical functions, optimisations, kinds of growth).

(b) It must be much clearer that central mathematical concepts of the lower secondary level are also good as standard models. This holds in particular for the concept of function: $y = c \cdot x$ and $y = c : x$ and *Schlussrechnung* (calculation with ratios), $y = a \cdot x + b$ and costs for many services, quadratic functions and uniformly accelerated movement, exponential function and geometrical growth, trigonometrical functions and periodical processes, and 'hidden' functions such as formulas for areas in geometry, the mean in statistics, the uniform distribution in probability theory.

(c) Students' everyday life and their fields of activity as well as other school subjects must be taken into consideration in an appropriate way. This does not exclude the preference of areas in which the teacher has specific competencies and (or) in which the students are especially interested.

(d) Students should regard themselves as 'wanderers between two worlds' reality and mathematics, and appreciate the importance of knowledge and understanding in both worlds in order to cope with real-world problems.

(e) For this appreciation it is necessary deliberately to change the viewpoint during the process of application, not only in the ordinary manner, applying a piece of mathematics to a well-prepared piece of reality, but also by noticing that there may be several fitting models for the same situation, that they now have to combine parts of mathematics rather isolated till now, that they first have to construct the mathematical means which are helpful in the given situation, and so on.

(f) There seem to be no insurmountable difficulties in introducing the modelling process according to Fig. 5.2 rather early. The appertaining steps may be introduced as reactions to heuristical questions such as 'Did you really answer the question?', 'Is this a fitting answer (a fitting question!)?', 'Can we improve the solution?', 'What don't we know yet?' (see Schönbeck & Schupp 1976).

(g) A deeper understanding of the relations between situation and model yet requires some maturity and the ability to think in relative and evaluating terms, which the students acquire only by the end of lower secondary level. Thus the teacher should not plan special units for the reflection of these relations in advance, but be prepared when the corresponding questions emerge by themselves while the students discuss a knotty problem.

(h) At this moment I am not able to give a definite statement about the role of the computer in our field. As a tool it offers considerable help, but its contributions to the difficult phases of the control circuit are rather modest.

4. CONCLUSION

If one fights for applied mathematics teaching, one can actually promise only 'blood, sweat and tears' and, moreover, must already fear one more wave (the computer wave) which will be supported as widely and extensively as that first.

However, I am not without hope. Relation to the real world is not a wave, but a permanent and indisputable claim, with which at present – fortunately – more and more teachers feel involved.

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Section A

Theory and Methodology

6

Mathematics and Physics: Where is the Difference?

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SUMMARY

Now, mathematics and physics appear as two distinct domains of knowledge. However if these two domains are actually distinct, the frontier that separates them is blurred: on the one hand, physics played (and still plays) a great role in the development of mathematical concepts and theories, on the other hand, the place of mathematics is close to the elaboration of physical concepts, and more generally of physical thinking.

Mathematical physics (of which elementary geometry is a part) is both mathematics and physics and it is not easy to define the part of each of these two domains in the building of mathematical physics. An example are the *Elements* of Euclid, the first model of a mathematised physical theory.

This consideration poses the problem of teaching of these two domains of science and their relations; also it is necessary that mathematics teachers speak about physics and physics teachers speak mathematics; particularly in the beginning of scientific teaching. I think the two domains must not be separated.

La distinction entre l'abstrait et l'expérimental n'est que de tendances, non d'essence.
Ferdinand Gonseth, Les Fondements des Mathematiques

Nowadays, mathematics and physics appear to be two distinct domains of knowledge. However if these domains are indeed distinct, the frontier that separates them is blurred; on the one hand, physics has played (and still plays) a fruitful role in the development of mathematical concepts and theories, while, on the other hand, mathematics are not only a tool for physicists but are also very close to the development of physical concepts and, even, of physical thinking.

Unfortunately this close relationship is not always emphasised in teaching where mathematics is proposed to physicists exclusively as a tool and physics is proposed to

students of mathematics as a series of applications of mathematical theories. In this narrow perspective, the problem of the use of mathematics (mathematics as a *service subject*) becomes artificial; mathematics is deprived of its actual and historical connection with scientific practice.

It is therefore necessary to identify the relationship between these two domains; a historical and epistemological approach leads to an awareness of this relationship and we must try to develop it in our teaching of mathematics and physics.

1. DERIVATIVES AND SPEEDS

What does the definition of speed as the derivative of displacement with respect to time mean? Is it merely a computational procedure as a high school student once told me, a conventional procedure to solve problems?

If a student has not understood the concept or derivative (even if he knows how to compute it), can he understand this definition of speed?

The problem is not to give a good definition of speed of derivative. Derivative and speed are two very closely related concepts and the understanding of any one is a step towards understanding the other; the problem is less one of giving a formal definition than of seeing how we use this idea in problems. Without such problems, the formal definition is meaningless.

Historically the concept of derivative was born in the study of motion and only later did it achieve independence from mechanics. Nevertheless this definition from motion not only remains relevant to mechanics but it also permits us to use meaningful mechanical metaphors in other problems where derivatives are used, such as the study of variations of functions of one (not necessarily real) variable, the study and the construction of curves (where the tangent vector is like a speed and the second tangent vector is like an acceleration), and so on.

2. VECTOR CALCULUS, VECTOR ANALYSIS AND LINEAR ALGEBRA

In France, with the introduction of the *new mathematics* introduced in teaching in 1970, the notion of vector disappeared behind the notions of vector space and affine structure. As a result of this, we encountered many difficulties in the teaching of physics, particularly in the study of speed and force, where using the classical notion of *oriented segment* is required.

Vectors (defined as oriented segments) are a good representation for speeds and forces, but there is no need to know vector theory in order to represent speeds and forces; on the contrary vector calculus is an *a posteriori* means of unifying these representations; vector calculus is a means of obtaining simultaneously the properties of speeds and forces (thus Appell introduced a preliminary chapter on vectors in his *Traité de Mécanique rationnelle* (1893)). However vector calculus is not just a tool we use in mechanics, it is very close to the formulation of mechanics; calculus on speeds and calculus of forces are *already* a vector calculus, but a calculus on mechanical objects, a calculus of which the vector calculus is an *abstract* representation.

Of course, vector calculus is not reduced to its mechanical meaning, but the understanding of vector calculus and its use in mechanics depends upon this unification of mechanical concepts (speeds, forces) and geometrical concepts (oriented segments), and

not upon an application of an *abstract vector calculus* to mechanics. It is remarkable that the same calculus is relevant for speeds and forces and also for translations, and remarking this contributes towards an understanding of the use of vector calculus.

We encounter the same problem with the concepts of vector analysis (gradient, divergence, curl) which are close to field theory. The problem is less to define these concepts than to show how problems of mechanics, electricity or magnetism, lead to these concepts (for instance, fields and potentials, divergence and conservation laws).

Thus vector analysis is not merely a tool for field theory, it is constructed within field theory; when vector-analysis becomes independent of field theory, it becomes a metaphoric language whose pertinence is based on its original physical meaning (such as the mathematical theory of potential). Thus, vector analysis is not a simple exercise in analysis; mathematical constructions and physical concepts must be understood simultaneously, each depending upon the other, each allowing the understanding of the other.

Another problem is posed by linear algebra. Histories of linear algebra and vector calculus are different, the first time the two appeared together was in *Calcolo geometrico* by Giuseppe Peano in 1888, and later, in 1918, Hermann Weyl published *Raum, Zeit, Materie* (*Space, Time, Matter*) showing how linear algebra is used in geometry and in physics (Cousquer, Crowe).

Vector calculus in elementary geometry and in elementary mechanics is independent of linear algebra, even though it will eventually become a mere chapter of linear algebra as Dieudonne argues in his *Algèbre linéaire et Géométrie élémentaire*; on the contrary it is through the several situations where linearity appears, in geometry, in physics, in analysis, that linear algebra takes its meaning as a method for unifying different concepts. It is precisely because it is a method of unification that linear algebra is so powerful, but its power can only be understood if we know what concepts linear algebra unifies, concepts which are not, in themselves, concepts of linear algebra.

3. ELEMENTARY GEOMETRY

Geometry is both a part of mathematics and a part of physics. In his paper *Geometrie und Erfahrung* (*Geometry and Experience*) (1921), Albert Einstein explains the distinction between the *mathematical geometry* defined by an axiomatical structure and the *practical geometry* which is closed to the study of space and therefore is the oldest part of physics. However this distinction became meaningful only in the nineteenth century, after the discovery of non Euclidean geometry, the structural viewpoint with the introduction of group theory in geometry by Felix Klein in the Erlangen Programme, and the axiomatical construction of geometry by David Hilbert. Before this, there was only one geometry, the one what we call today *elementary geometry* in which the distinction between a mathematical part and a physical part is meaningless. This geometry is the science of spatial configurations that was presented by Euclid in the *Elements*; this geometry deals with the measurement of geometrical magnitudes (lengths, areas, volumes, angles) and it is based on the *principle of equality by coincidence* (fourth axiom in Heath's edition of the *Elements*) that states that

Things which coincide with one another are equal

This principle is very close to the motion of solid bodies as explained by the French mathematician Jules Houel in his *Essai critique sur les principles fondamentaux de la géométrie élémentaire* (1867). By this principle, elementary geometry allows a development of classical mechanics of solid bodies which shows the pertinence of Euclid's geometry for a knowledge of the physical world (Enriques, Gonseth).

However if geometry is a preliminary to mechanics, mechanics plays a role in the development of geometry as Isaac Newton explained in the Preface to the *Principia.*

> *Geometry does not teach us to draw these lines* (straight lines and circles), *but requires them to be drawn, for it requires that the learner should first be taught to describe these accurately before he enters upon geometry, then it shows how by these operations problems may be solved. To describe right lines and circles are problems, but not geometrical problems. The solution of these problems is required from mechanics, and by geometry the use of them, when be solved, is shown. . . Therefore geometry is founded in mechanical practice, and is nothing but that part of universal mechanics which accurately proposes and demonstrates the art of measuring.*

Thus to the second point of the Euclidean corpus: the *instrumental character of geometry* is specified by the use of ruler and compass to solve problems of constructions; ruler and compass are both theoretical and practical instruments, both to define and to draw straight lines and circles.

These two points of Euclidean geometry, the principle of equality by coincidence and the instrumental character, are close to the empirical origin of geometry. Thus, elementary geometry as it was developed by the Greek geometers and continued by their successors, is the model of a rational science built on empirical data, the first example of mathematical physics, that is to say, a means of knowing the world by the deductive method. The deductive method is less a logical problem (the technical side of mathematics) than a metaphysical (or theological) problem: to show the necessity of the truths of Science (Aristotle). That problem of necessity remains an ideal for mathematical physics today, even if the problem is less one of truth than one of internal coherence and of consistance with experiments, the *idonéité* to use a term from the Swiss philosopher of mathematics Ferdinand Gonseth, one of the greatest of our century.

Another great problem that played a role in the development of geometry, is posed by the representation of space, that is to say, drawing a spatial object on a plane. This problem was studied by the painters and architects of the Italian Renaissance, and led towards the development of linear perspective and projective geometry. I shall add mention that the works from which modern projective geometry was created deal with a military problem, the building of fortifications; these works led towards the descriptive geometry of Gaspard Monge, at the end of the eighteenth century. Monge, who was a mathematician, understood the dual nature of descriptive geometry: a method of drawing and a method of study in rational geometry (Chasles, Monge). However I cannot speak about this in detail here and I refer you to the recent book by Roger Laurent and Jeanne Peiffer: *La place de Lambert dans l'histoire de la perspective.*

To conclude this discussion about geometry, let me recall an important aspect of contemporary mathematics and physics: the *geometrisation*, which Dieudonne has called the *universal domination of geometry*. Geometrisation of which a first example is given in the arithmetical books of the *Elements* of Euclid, appears as a metaphoric